Analytic Method for Probabilistic Cost

and Schedule Risk Analysis

Final Report

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1 Executive Summary

Estimates of cost and schedule duration of a task or project are uncertain values, so we do not know the exact, discrete values until it is complete. Given the inherent uncertainty of estimates, the only way to portray them is with probability distributions of possible costs and schedule durations (or dates). Probabilistic cost and schedule distributions for a program are quantified through the means of cost and schedule uncertainty analyses. The most popular way these analyses are performed is though statistical simulation. Statistical simulation (i.e., Monte Carlo and Latin Hypercube sampling) techniques are widely used in cost and schedule risk analysis, but they have limitations.

Analytic methods of cost and schedule risk analysis exist that: 1) correctly model random variables (RVs); 2) exactly correlate RVs and their sums, which many statistical simulation tools cannot; 3) have no fundamental limit to the number of RVs or correlation coefficients that can be defined; 4) provide [near] instantaneous results; and 5) have the ability due to their mathematical form to clearly indicate uncertainty drivers and thus the risk.

This report presents an analytic (i.e., a non-simulation based) method of quantitative cost and schedule risk analysis building on analytic techniques of applied probability and statistics. The analytic method provides near-instantaneous results with exact statistics such as mean and variance of total cost and total schedule duration. It capitalizes on the fact that the structure of estimates defines a mathematical problem to be solved through the use of applied probability. This report provides the mathematics required to perform the tasks of calculating the uncertainty of an estimate, and determining the risk from this uncertainty and a point estimate.

While much of the mathematics of applied probability used in this report are publicly available through journal publications, the author has derived methods and formulae that have, to his knowledge and through his research, never been published before. Therefore, the report provides a very unique set of mathematics useful in the analytic assessment of cost and schedule uncertainty and risk.

The report includes several quantitative examples, including two example estimates, where the results obtained using the analytic method compare well with those results obtained through statistical simulation. Given the excellent results obtained through this research, additional applications of the analytic method are recommended for use in risk analysis, estimating relationship development, and probabilistic cost and schedule estimating.

2 Introduction

This report describes an analytic method of applied probability analysis techniques germane to problems encountered in cost and schedule risk estimation. By their very nature, estimates are uncertain projections of future events. Given that, we discuss the probabilistic nature of estimates and describe the mathematical problems encountered in cost and schedule estimating. We discuss the mathematical tools that can be used to solve these problems (i.e., statistical simulation and statistical analysis) and we compare the two approaches. The next sections of the report provide the tools required to perform statistical analysis. Finally, we provide two sample problems to demonstrate analytical techniques.

2.1 Probabilistic Nature of Estimates

Cost and schedule estimating is an integral part of the program management process. Organizations use these estimates for planning purposes such as cost/performance tradeoff studies, benefit/cost analyses, source selections, and budget planning. But estimates are predictions and their exact values are uncertain in nature since they have not yet become "fact". Since the true cost and schedule durations of a project (or task) are only known when it is complete, the best we can do is to rely on estimates at various stages of planning and completion.

The word "estimate" itself implies uncertainty, so an estimate is not well represented by a single number but by a distribution of possible estimates. The distribution of possible estimates is defined by the estimate's probability distribution that is calculated through the application of probability and statistics.

2.2 Uncertainty and Risk

Uncertainty is a measure of the distribution of possible outcomes of a random variable, such as cost and schedule estimates. This distribution is called a probability distribution and can either be a continuous, discrete, or mixed distribution.[1]

2.2.1 Probability Density and Probability Mass

Probability distributions defined for continuous distributions are probability density functions (PDFs). PDFs such as the one shown in Figure 2-1 can be expressed in terms of a mathematical formula of $f_X(x)$, where $f_X(x)$ is the PDF defined over the range, x.

[1] A "mixed distribution" is a combination of discrete and continuous distributions.

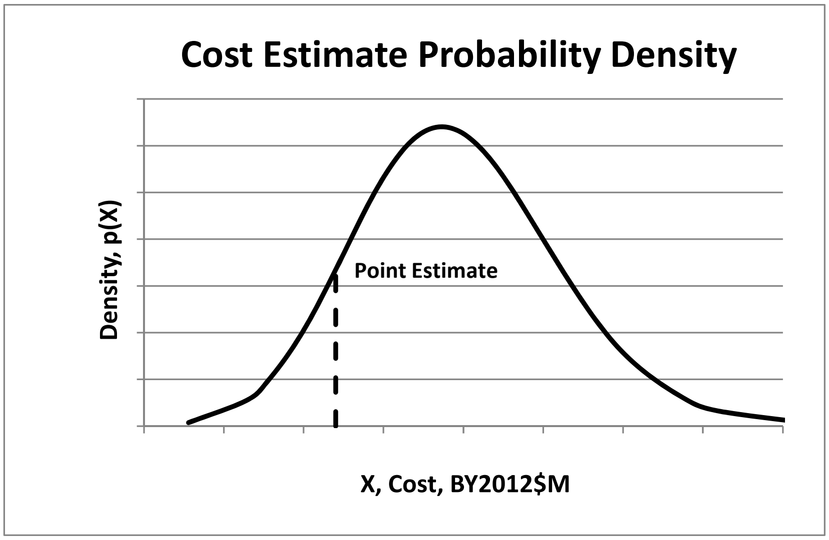

Figure 2-1 Probability Density Distribution

Probability distributions of discrete risks (which are discontinuous functions) are defined by probability mass functions (PMFs) such as the one shown in Figure 2-2. We will define the PMF as $g_X(x)$, where $g_X(x)$ is the function defined over the range x.

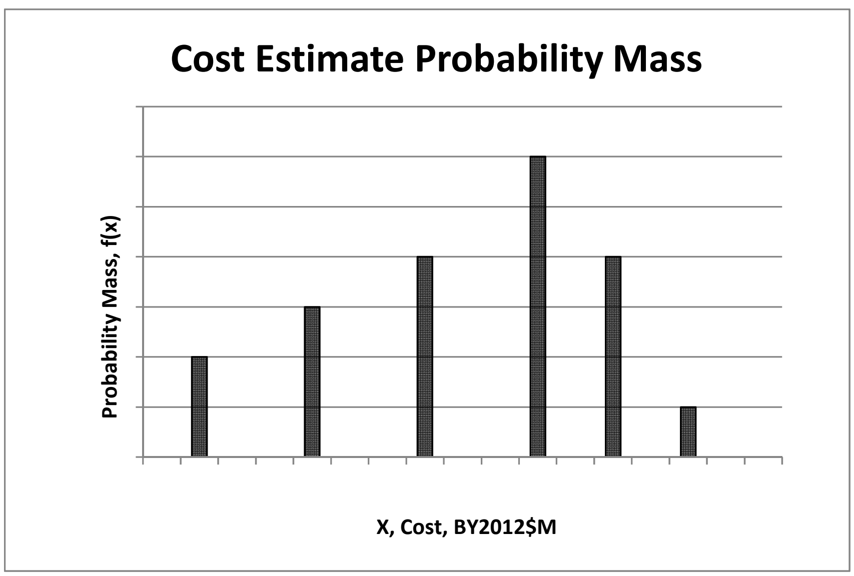

Figure 2-2 Probability Mass Distribution

2.2.2 Cumulative Probability

The cumulative probability is the probability that a real valued random number will be less than some value x. In the case of discrete distributions, it is the sum of the probability-weighted values of the PMF less than x, and in the case of continuous distributions, (remembering our college calculus) it is the integral of the PDF from $-\infty$ to x.

2.2.3 Definition of Risk

Any point estimate has some probability that it will be sufficient or be exceeded (Figure 2-3). The probability that an estimate will be exceeded (i.e., overrun) is the risk, and the probability that the estimate will be sufficient (and that there is a probability of the actual value being lower) is the opportunity or reward.

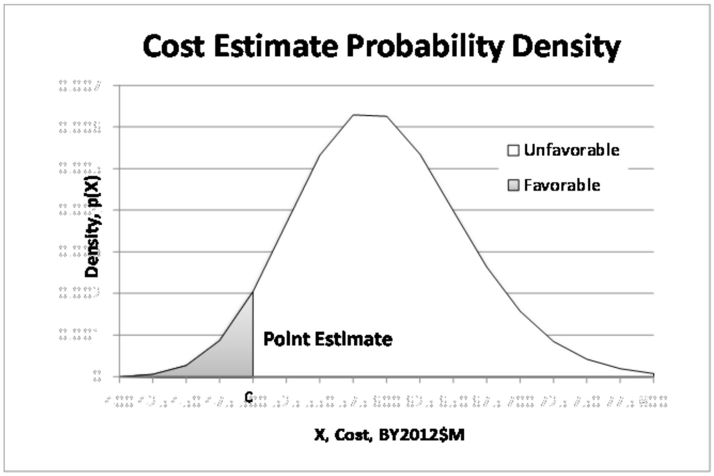

Figure 2-3 Risk, Reward and the Point Estimate

Since the entire area under the PDF shown in Figure 2-3 is, by definition, equal to one, the sum of the probabilities of overrun (risk) and under-run (reward or opportunity) is also equal to one. The probability of risk occurrence is the area of the distribution to the right of the point estimate and the probability of reward is the area to the left. As stated earlier, the area of the distribution under a curve can be computed using the definite integral expression bounded by the lower and upper limits. Therefore, risk is the integral of the PDF from the point estimate, c, to infinity (∞).

$$Risk = \int_c^{\infty} f_X(x)dx = 1 - \int_{-\infty}^{c} f_X(x)dx = 1 - F_X(c). \quad \textbf{2-1}$$

Reward or opportunity represents the area under the curve from $-\infty$ to c, which is

$$Reward = \int_{-\infty}^{c} f_X(x)dx = F_X(c). \quad \textbf{2-2}$$

If we are using discrete risks defined by PMFs, then the risk equation is a summation of all of the probability-weighted risk consequences at all points x (i.e., costs or schedule durations) (Garvey P. R., 2000) greater than our point estimate, c.[2]

$$Risk = \sum_{x>c} P_X(x). \quad \textbf{2-3}$$

[2] Garvey, P. R. (2000). Probability Methods for Cost Uncertainty Analysis: A Systems Engineering Perspective. New York, NY: Marcel Dekker.

The amount of risk to an estimate is defined by two things: the uncertainty of the estimate and the point estimate, or the bet. To illustrate the interaction of risk with uncertainty and the bet, consider the four examples in Figure 2-4. Figure 2-4a. is a low-uncertainty, high-risk estimate since the area under the PDF to the right of the bet is much larger than that to the left. This means there is a disproportionate amount of risk compared to opportunity. In in Figure 2-4b, the risk is reduced by choosing a bet further to the right in the PDF. Note that in both of these cases, the potential low- and high-end outcomes remain the same – only the bet is changed. When the low bet is accompanied by a larger estimate uncertainty, as in in Figure 2-4c, the risk is reduced, but the potential impacts due to high-end outcomes (consequences) are increased. Finally, moving the bet to the right in the high uncertainty case, the risk is reduced as shown in in Figure 2-4d, but the potential for extreme high-end outcomes remains.

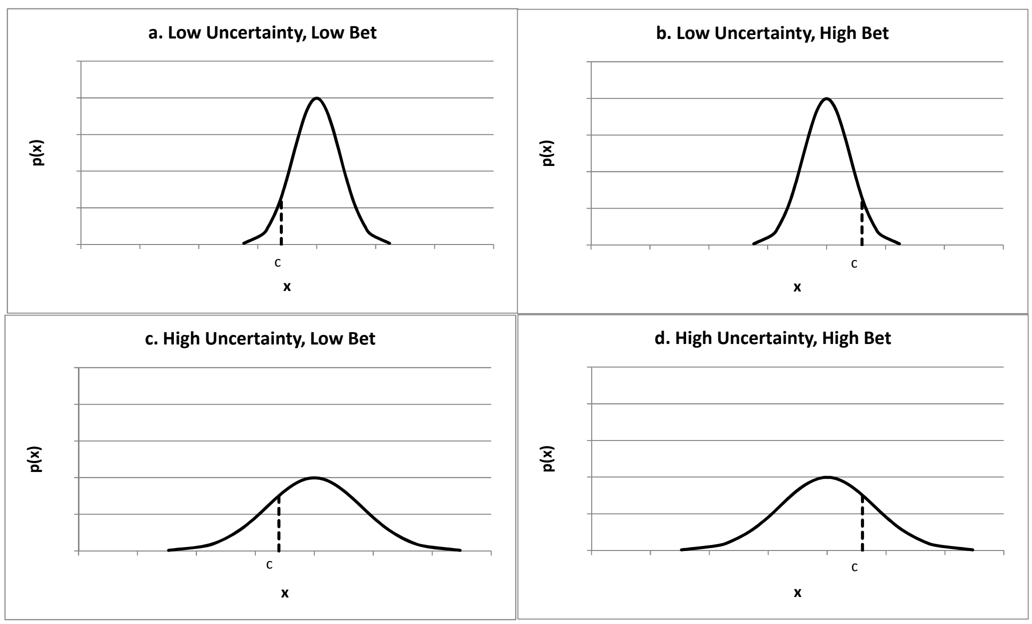

Figure 2-4 Relationship between Risk, Uncertainty and the Bet

2.3 Joint Probability Distributions

So far we have discussed the univariate[3] probability distributions of single random variables (i.e., estimates of cost or schedule). When we are interested in the probability distribution of more than one random variable, we are interested in the multivariate probability distributions, such as the probability of achieving a particular cost and schedule of a yet-to-be-completed project. When the relationships between variables such as estimated cost and schedule must be considered, we need to form a joint probability distribution. An example of this is shown in Figure 2-5.

[3] Single variable

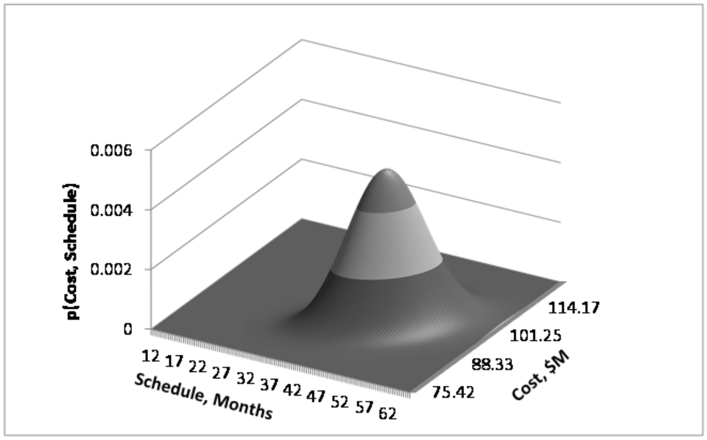

Figure 2-5 Joint Probability Density Function

If we have two random variables X and Y, we can define the probabilities

$$P\{X \leq x\} = F_X(x) = \int_{-\infty}^{x} F_X(z)\, dz \qquad \textbf{2-4}$$
$$P\{Y \leq y\} = F_Y(y) = \int_{-\infty}^{y} F_Y(z)\, dz$$

The joint probabilities of $P\{X \leq x, Y \leq y\}$ can be expressed as the joint distribution function

$$P\{X \leq x, Y \leq y\} = F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(z, w)\, dzdw \qquad \textbf{2-5}$$

The joint PDF is defined as the partial derivative of $F_{XY}(x, y)$ with respect to x and y.

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \qquad \textbf{2-6}$$

2.3.1 Marginal Distributions

The marginal distributions of a joint probability function are those distributions that are considered individually. Given a joint distribution of two random variables, the marginal distribution of one is its probability distribution averaged over the probability information from the other's distribution.

2.3.2 Conditional Distributions

A conditional distribution of a joint probability function is the distribution of one random variable given a specific value of the other distribution(s).

2.4 Statistics of a Random Variable

2.4.1 Moments

Moments provide useful information about the characteristics of a random variable, X, such as the measures of central tendency, dispersion and shape. When referring to the moments of a distribution or a set of data, it is useful to define which of the three types of moments are being used: raw moments, central moments or standardized moments.

2.4.1.1 Raw Moments

The k^{th} moments about the origin are called "raw moments" of a PDF, f_X, and are defined as:

$$\mu'_k = \begin{cases} \sum_X x^k f(x) & ; if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^k f(x)dx & ; if\ X\ is\ continuous \end{cases} \qquad \textbf{2-7}$$

The mean, μ'_1 , is the first raw moment of X about the origin, and it is a measurement of the central tendency of the data. We are more familiar with the mean being represented as, μ, so we will use this notation for the mean hereafter.

2.4.1.2 Central Moments

Central moments of a distribution are the raw moments about the mean, μ. The first central moment is by definition zero, but the second central moment is the variance, σ^2, which is a measure of dispersion about μ. Equation 2-8 provides the definition of the k^{th} central moments of discrete and continuous RVs.

$$\sigma^2 = \begin{cases} \sum_X (x-\mu)^2 f(x) & ; if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx & ; if\ X\ is\ continuous \end{cases} \qquad \textbf{2-8}$$

The variance, σ^2, is the square of the standard deviation, σ.

The first five *central moments* expressed in terms of the raw moments are:

$$\mu_1 = 0 \qquad \textbf{2-9}$$
$$\mu_2 = -{\mu'_1}^2 + \mu'_2 = \mu'_2 - {\mu'_1}^2 \qquad \textbf{2-10}$$
$$\mu_3 = 2{\mu'_1}^3 - 3\mu'_1\mu'_2 + \mu'_3 \qquad \textbf{2-11}$$
$$\mu_4 = -3{\mu'_1}^4 + 6{\mu'_1}^2\mu'_2 - 4\mu'_1\mu'_3 + \mu'_4 \qquad \textbf{2-12}$$
$$\mu_5 = 4{\mu'_1}^5 - 10{\mu'_1}^3\mu'_2 + 10{\mu'_1}^2\mu'_3 - 5\mu'_1\mu'_4 + \mu'_5 \qquad \textbf{2-13}$$

2.4.1.3 Standardized moments

Standardized moments are the k^{th} central moments, μ_k, normalized by the k^{th} powers of the standard deviation σ^k (i.e., $\frac{\mu_k}{\sigma^k}$).

The most well-known standardized moments are skewness and kurtosis. Skewness, ϑ, is the measure of asymmetry of X and is defined as the third standardized moment:

$$skew(X) = \vartheta = \frac{\mu_3}{\sigma^3} \qquad \textbf{2-14}$$

A distribution is a) symmetric if $\vartheta = 0$, b) left (i.e. negatively) skewed if $\vartheta < 0$, and c) right (i.e., positively) skewed if $\vartheta > 0$ as shown in Figure 2-6.

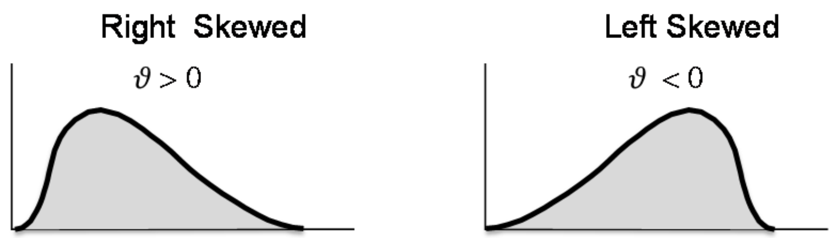

Figure 2-6 Left and Right Skewed Distributions

Kurtosis is the fourth standardized moment. Most textbooks define kurtosis of *symmetric, unimodal* distributions as a measure of peakedness of a distribution X. This is a correct definition, however a more descriptive definition of kurtosis exists (DeCarlo, 1997), (Moors, 1986), (Balanda & MacGillivray, 1988), and (Darlington, 1970).[4, 5, 6, 7] Moors defines kurtosis as the measure of the dispersion around the two "shoulders" of a distribution located at $\mu \pm \sigma$. DeCarlo warns that the classical attribution of peakedness of a distribution vice its "fat-tailedness" is not a good representation of the meaning of kurtosis and provides examples where this is the case.[8]

$$kurt(X) = \frac{\mu_4}{\sigma^4} \qquad \textbf{2-15}$$

A more commonly used metric is the "excess kurtosis", which is $kurt(X) - 3$. Since the kurtosis of a normal distribution is equal to three, the excess kurtosis denoted as κ, is adjusted by 3 as in Equation 2-16.

$$\kappa = kurt(X) - 3 = \frac{\mu_4}{\sigma^4} - 3 \qquad \textbf{2-16}$$

In general, where a) $\kappa = 0$ the distribution is mesokurtic, b) $\kappa > 0$ it is leptokurtic, and c) $\kappa < 0$ it is platykurtic.

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[4] DeCarlo, L. (1997). On the meaning and use of kurtosis. Psychological Methods, 292-307.
[5] Moors, J.J.A. The meaning of kurtosis: Darlington reexamined. *Amer. Statist*.**1986**, *40*, 283-284.
[6] Balanda, K.P.; MacGillivray, H.L. Kurtosis: A critical review. *Amer. Statist*.**1988**, *42*, 111-119.
[7] Richard B. Darlington. Is Kurtosis Really "Peakedness?". *Amer. Statist*. **1970**, *24*, 19-22.
[8] DeCarlo, L. (1997).

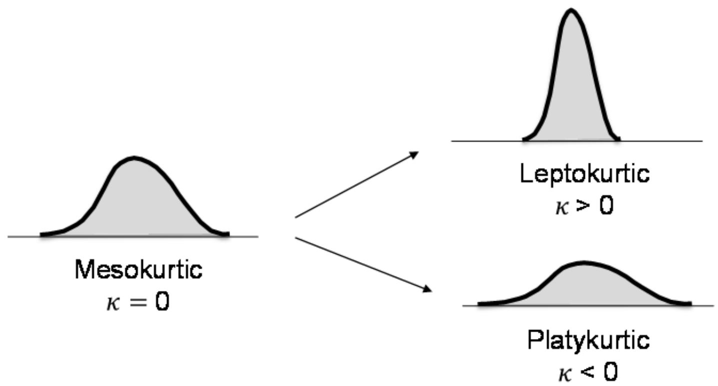

Figure 2-7 Excess Kurtosis of Distributions

2.4.1.4 Moment Summary

The moments describing the characteristics of a random variable such as the measures of central tendency, dispersion and shape (i.e., $\mu, \sigma^2, \vartheta, \kappa$) can be derived from the raw moments μ_k' of X. We will capitalize on these relationships in the analytic method proposed in this report.

2.4.2 Quantile Statistics

Quantiles are a set of divisions of data into groups containing equal numbers of observations. We are most familiar with percentiles, which are division of the data into 100 groups of 1% of the cumulative area under a PDF. We will denote the percentile, q, of a random variable, X, as $X_{z=q}$. For example the 50th percentile of X would be written $X_{z=0.5}$.

2.4.3 Expectation Operator

The expectation operator, $E[\cdot]$, of a random variable is a powerful expression. The expected value, or μ, (Equation 2-17) of a random variable is perhaps the most important single parameter in applied probability. It is written as

$$E[X] = \mu_X \ , \qquad \textbf{2-17}$$

and is the integral

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx, \text{ where } f_X(x) \text{ is the PDF of } X. \qquad \textbf{2-18}$$

The mean represents the center of gravity of the random variable. Another important parameter is σ^2, defined by the expectation of the squared difference of the PDF and its mean. This quantity represents the moment of inertia of the probability masses (Papoulis, 1965).[9]

$$Var(X) = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (X-\mu)^2 f_X(x)dx \qquad \textbf{2-19}$$

What is most important about $E[\cdot]$ is its ability to determine the raw moments (Equations 2-7 and 2-18) and central moments (Equations 2-8 and 2-19) of a random variable, and thus the measures of central tendency, dispersion and shape (i.e., $\mu, \sigma^2, \vartheta, \kappa$).

2.4.4 Order Statistics

Order statistics are those statistics that describe the numerical order in which random variables or samples of random variables appear. Some of the simplest order statistics are the minimum and maximum values defining the range of a PDF. Other, more complex order statistics are those which describe the maximum and minimum of a series of random variables. Order statistics play an especially important role in schedule risk analysis whereby the maximum probabilistic end dates of certain tasks define the maximum probable end-date of the schedule.

2.5 Section Summary

The mathematics of the analytic techniques used to solve estimating uncertainty problems require definition of the estimating problems germane to cost and schedule estimates. In the next section, we discuss the mathematical problems typically found in cost and schedule estimating.

9 Papoulis, A. (1965). Probability, Random Variables and Stochastic Processes. New York, NY: McGraw Hill.

3 Cost and Schedule Estimates

Cost and schedule estimates are defined by a set of mathematical formulae that lend themselves to probabilistic uncertainty analysis. In this section, we will discuss the structures of these types of estimates and define the mathematical problem(s) to be solved in probabilistic uncertainty analysis.

Book[10,11] (1994; 2002) showed the cost and schedule estimating communities that every cost and schedule estimating problem should be treated as a risk analysis, not simply an exercise in summing most likely costs – the result of which is a number that has no statistical meaning without risk analysis. Furthermore, he showed estimates should be treated as random variables and not deterministic numbers (i.e., constants).

3.1 Nomenclature

To better describe the mathematical problems germane to cost and schedule estimates, we will define constants, variables, and random variables.

A numerically expressed entity is called a "constant" if there is a unique specific number that is always its numerical value (e.g., π, 1.414, -2). A numerically expressed entity is called a "variable" if there are several possible specific numbers that may serve as its numerical value and which specific number *happens* to be its numerical value in any particular situation depends on the particular circumstances (e.g., x, y, z)[12]. A variable is further denoted a "random variable" if the proportion of particular situations in which any specific number happens to be its numerical value is established by a probability distribution (e.g., X, Y, cost, schedule duration).

We will use the following notation throughout this document to define variables. Constants will be defined using their numerical value or lowercase letter (e.g., a, b, c, d, e). Variables will use lowercase letters $u, v, w, x, y,$ and z, and random variables will use uppercase letters U, V, W, X, Y and Z. Random variables defined by commonly used PDFs will use the following notation:

Uniform:	$f_X(x; L, H) = U(L, H)$	**3-1**
Triangular:	$f_X(x; L, M, H) = T(L, M, H)$	**3-2**
Normal:	$f_X(x; \mu, \sigma) = N(\mu, \sigma)$	**3-3**
Lognormal:	$f_X(x; \mu, \sigma) = L(\mu, \sigma)$	**3-4**
Beta:	$f_X(x; \alpha, \beta, a, b) = B(\alpha, \beta, a, b)$	**3-5**

Where

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[10] Book, S. A., "Do Not Sum 'Most Likely' Cost Estimates", 1994 NASA Cost Estimating Symposium, Johnson Space Center, Houston, TX, 8-10 November 1994.

[11] Book, S. A., "Schedule Risk Analysis: Why It is Important and How to Do It", Ground Systems Architectures Workshop, The Aerospace Corporation, El Segundo, CA, 13-15 March 2002.

[12] Book, S. A., 1994.

L, M, H are low, most likely (mode), and high shape parameters
μ, σ are the mean and standard deviation of the distribution in unit space
α, β are standard beta distribution shape parameters
a, b are lower and upper bounds of the four-parameter beta distribution

The properties of these distributions are provided in Appendix A – Probability Distributions.

3.2 The Cost Estimating Problem

The cost estimating problem is defined by the mathematics of the following: 1) the work breakdown structure (WBS), which requires multiple levels of statistical summation; and 2) the mathematics most applicable to the estimating approach(es) used (i.e., bottom-up, analogy, parametric). We will first describe the statistical techniques used to perform statistical summation of a WBS structure and then discuss, in more depth, how to apply analytic uncertainty and risk analysis to the individual WBS elements.

3.2.1 WBS structure

The WBS defines the summation hierarchy of the project. In other words, it defines the mathematical problem of summation of individual WBS elements to successively higher levels of the WBS up to the total project level. The statistical treatment of summing correlated random variables is fairly straightforward and can be easily programmed into a spreadsheet or cost estimating tool (Young, 1992).[13]

3.2.2 Estimating Methods

The methods used to estimate costs at different WBS levels define another part of the mathematical problem to be solved. Different estimating methods require different mathematical procedures, so we will examine these methods individually and note the important mathematical features of each. These include bottom-up, analogy approach relying on scaled actuals, multiple scaled actuals, and cost estimating relationships (CERs).

3.2.2.1 Bottom-up

The bottom-up estimating approach relies on summing a detailed list of the classical elements of cost: labor (effort), material and expenses. If a detailed, resource-loaded schedule is used to estimate effort, then the duration of the task, the staffing level and the associated labor rates can be represented by random variables. As an example, the cost of

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[13] The “Formal Risk Assessment of System Cost Estimates” (FRISK) method is an analytic risk model that uses “Method of Moments” to calculate summary distributions. FRISK was originally developed by Phil Young of The Aerospace Corporation in 1992 (before Crystal Ball and @Risk became available) with funding from USAF SMC. A BASIC Program implementing FRISK was developed by Dr. Stephen Book and enjoyed many years of use. FRISK has been reprogrammed in Excel by various analysts since 2000, with each new version providing more advanced capability and features and ease of use.

the effort for a particular task is the product of the task duration, the resource loading profile and the associated labor rates. Each is treated as a random variable.

$W = XYZ$; where
W = effort, measured in dollars
X = duration of the task, measured in hours
Y = resource loading, measured in heads
Z = the labor rate, measured in dollars per hour per head

In this case, the first mathematical problem to be solved is the multiplication of multiple (and perhaps correlated) random variables. This will be discussed in Section 5. The second problem is the summation of the elements of cost represented by random variables for each WBS element, as discussed in Section 4.2.2.

3.2.2.2 Analogy (Scaled Actuals)

The analogy method relies on using an actual cost of a product or service to estimate the cost of a similar product or service. Intuitively, it is the easiest method to use when preparing a cost estimate. The simplest form of an analogy estimate is a direct analogy, in which case the estimated cost is equated to the actual cost of the similar product or service. Unfortunately, this simple procedure does not provide any information about the uncertainty of the estimate. Indeed, the analogy can be the most misleading estimating method from a probability perspective.

Studies (MacKenzie & Addison, 2000) by the Space Systems Cost Analysis Group (SSCAG) have shown the standard deviation of the costs of similar items at the "box level" of the WBS to be as much as 30% to 40%.[14] In the same report, the authors showed the data to be lognormally distributed, which provides a shape to the distribution. Given this information, we are able to derive a measure of the standard deviation of the "actual" cost based on the coefficient of variation ($CV = \mu/\sigma$), but we do not know at which percentile to place our particular analogy. Is it at the 50th percentile (median), the mode, the mean (expected value), or is it at some other percentile such as the 4th or the 85th, or somewhere else? If it is at the mean, then the PDF of the analog is easily determined. But, is this the right PDF to use in this situation? Figure 3-1 shows an example lognormal distribution based on the mean and $CV = 0.3$, $L(100, 30)$.

[14] MacKenzie, D. and Addison, B., "Space System Cost Variance and Estimating Uncertainty", 70th SSCAG Meeting, Boeing Training Center, Tukwila, WA, October 12-13, 2000.

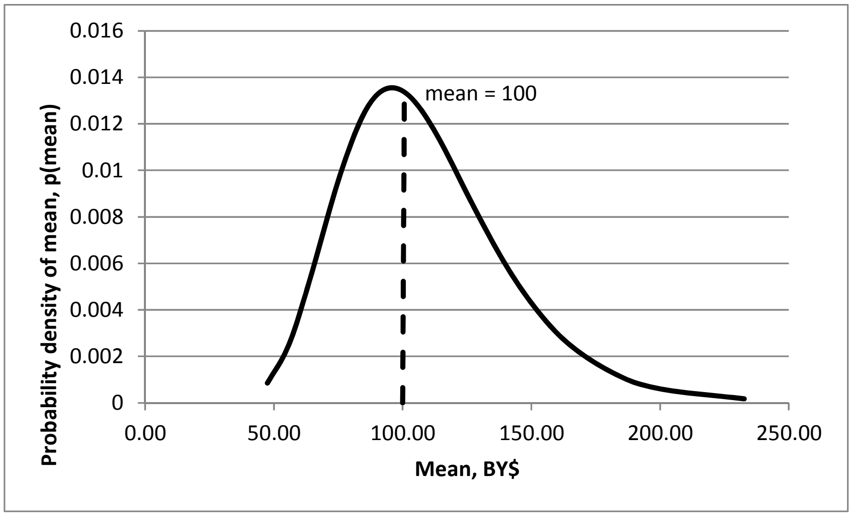

Figure 3-1 PDF of Cost of Analogy at Mean

Now consider the case where the analogy is one cost of many possible costs within an unknown probabilistic range. To provide a distribution about the analogous cost, we need to either 1) assume a percentile value for the analogy within a prescribed distribution, or 2) determine the (yet unknown) probabilistic range of possible values to which the analogous cost belongs. The first case is described by Flynn, Braxton, Garvey and Lee (2012).[15] The second case requires the use of applied probability to determine the probability distribution. The derivation for this approach is provided in Appendix C – Derivations.

3.2.2.3 Scaled Actuals (Factor)

If a simple factor is used to scale an actual cost, then the mathematical problem is the multiplication of random variables, where one random variable is the scaling factor and the other is the PDF of the analogy, described in Section 3.2.2.2.

3.2.2.4 Scaled Actuals (Interpolation)

When we estimate the cost of an item through linear interpolation of two actuals using a cost driver (i.e., weight), the mathematical problem is a linear relationship:

$$Y_e = Y_1 + (X_e - x_1) * \frac{(Y_2 - Y_1)}{(x_2 - x_1)}, \text{ where} \quad \textbf{3-6}$$

Y_e= the cost estimate (random variable)
X_e = the cost driver of the item we are estimating (a random variable)
Y_1, Y_2 = the costs of the two actuals, (random variable)
x_1, x_2 = the cost drivers of the two actuals (constant)

[15] Flynn, B., Braxton, P., Garvey, P., & Lee, R. (2012). Enhanced Scenario-Based Method for Cost Risk Analysis: Theory, Application and Implementation. 2012 SCEA/ISPA Joint Annual Conference & Training Workshop. Orlando, FL.

The plot of the discrete interpolation problem is shown in Figure 3-2.

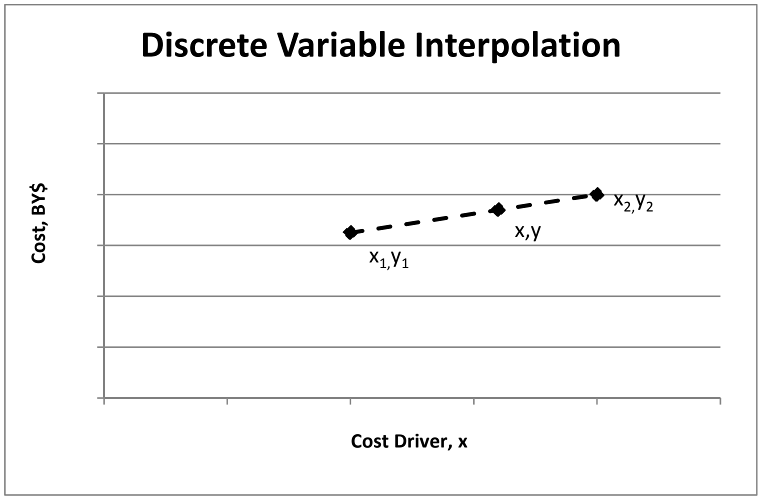

Figure 3-2 Discrete Variable Interpolation

The mathematical problems to be solved in Equation 3-6 are the addition, subtraction and multiplication of random variables.

Note the costs of the two actuals have a similar issue as the direct analogy method whereby we cannot assume the *a priori* standard deviations of the samples. If we cannot treat these samples of actual values as constants (no error) in the direct analogy case, then we shouldn't treat them as such in the interpolation case.

3.2.2.5 Multiple Scaled Actuals and Cost Estimating Relationships

Multiple scaled actuals are those actuals that are similar in nature and whose costs can be represented by a probability distribution or by simple moments such as μ and σ. For example, the costs of three-meter ground station antennas could be represented by a normal distribution, $N(\mu, \sigma)$. Provided the antenna of interest fits into the set of three-meter ground station antennas represented by the PDF, we know the μ, σ, and confidence level of each estimate in the range of the PDF.

When we are estimating costs of products or services that are based on a similar set of parameters, we can develop a cost estimating relationship (CER) that explains some of the variations in cost based on variations in one or more independent variables (i.e., cost drivers). Consider the generic form of a recurring CER based on unit theory shown in Equation 3-7.

$$y = \left\{\left[a + b \sum_{i=F}^{L}(u_i{}^{c}) \prod_{j=1}^{N}\left(x_j{}^{d_j}\right) \prod_{k=1}^{M}(e_k{}^{s_k})\right]\right\}\varepsilon \text{ ; where} \qquad \textbf{3-7}$$

$a, b, c, d,$ and e are coefficients of the regression ($c = ln_2(LCS_C)$),
LCS_C = cumulative average learning curve slope when a = 0,
u_i = unit number i,

x_j = independent variable j,
N = number of independent variables,
s_k = indicator ("dummy") variable k,
M = number of indicator variables, and
ε = percent standard error (multiplicative).

The independent variables, x_j, can be represented by random variables X_j as can the multiplicative error of the estimate, ε. The dependent variable, y, will also be a random variable, Y, defined by the PDFs of each independent variable, the functional transformation of the CER form, and the PDF of the multiplicative error, ε.

The CER provides a model for constructing the PDF, so we can obtain the μ, σ, and confidence level of each estimate in the range of the PDF as in the case of multiple scaled actuals. To compute the statistics of the CER, we must first learn how to convolve and transform random variables. This is discussed in Sections 4 through 7.

3.2.3 Discrete Risks

Analysts may need to include discrete risk events form a risk register (Table 3-1) in a cost or schedule estimate. In the single risk case, this means there is a probability that some estimate of additional cost or schedule will be added. With multiple risks, the problem becomes combinatoric, since we must account for any combination of risks that could potentially occur.

Historical cost and schedule actuals contain realized risks which may or may not have been mitigated or manifested themselves into cost and schedule growth from the original proposed estimate. By using historical actuals to form the estimating relationships, the resulting estimate 1) will appear more conservative than if it had been developed using engineering judgment or non-metric-based approaches; 2) will inherently contain schedule and cost risks typical of similar programs; and 3) will be more prone to double or even triple-counting risks when augmented with discrete cost and schedule risks from a risk register (Table 3-1).

Table 3-1 Example Risk Register

Risk ID	Description	Probability	Impact	Impact Area
R1	Additional program management personnel	0.50	$200,000	Cost
R2	Redesign of computer board	0.25	6 Months $75,000	Schedule Cost
R3	Parts failure	0.10	$250,000	Cost Technical
R4	Second vendor required	0.05	12 months	Schedule Technical
O1	Renegotiate subcontract	0.25	$100,000	Cost

To form a complete risk picture, additional cost-related risks identified by the schedule risk assessment (SRA) and the discrete risk analysis obtained from the risk and opportunities register (ROR) are included to form the risk profile of the program. In many cases, the historical risk inherent in the use of estimating methods developed from actual data covers many potential risks (Figure 3-3). In these cases, the analyst must identify unique risks and omit redundant risks (B and C) identified and represented in the SRA and ROR. The use of more robust statistical and risk analyses minimizes the unidentified and untracked risks (A).

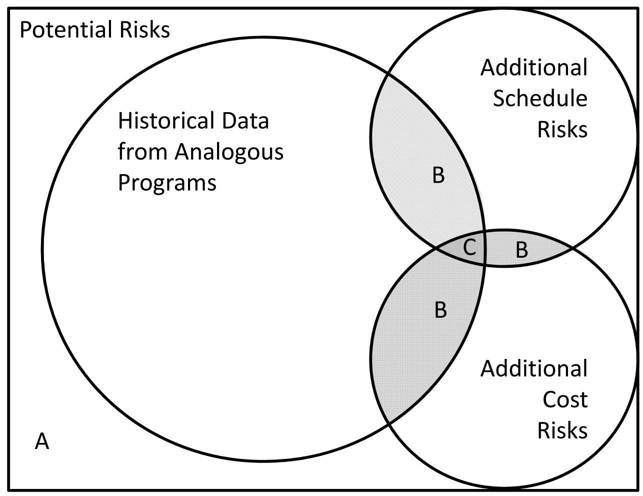

Figure 3-3 Estimating Risk Venn Diagram

3.3 The Schedule Estimating Problem

The schedule estimating problem is defined by the method used to estimate the schedule duration. When scaled analogy or multiple scaled actuals or schedule estimating relationships (SERs) are used to estimate schedule duration, the mathematical problem to be solved is similar to those of cost estimating. The two fundamental differences are: 1) probabilistic durations are measured in workdays, and 2) when the bottom-up approach is used, the schedule network defines the mathematical problem to be solved. We will discuss the issues that arise when using workdays rather than calendar days and then discuss the issues arising from the arrangement of tasks in a network.

3.3.1 Using Workdays in a Schedule

When using workdays in a program schedule, probabilistic dates are expressed as discrete rather than continuous distributions. This arises from the fact that a particular task may finish on a particular day (or part of a work day) but not all possible values within the range. Consider the example of the duration of a task to be a continuous, uniform distribution defined as $U(1,2)$. The lower bound of the continuous distribution is defined as one day and the upper bound as two days. Assuming a continuous distribution for the duration of the task, the finish date of the task will be within the range of one to two days later. In our example, the mean and standard deviation of the duration's continuous uniform $U(L,H)$ distribution are:

$$\mu_{U(1,2)} = \frac{L+H}{2} = 1.5 \text{ days}$$

$$\sigma_{U(1,2)} = \sqrt{\frac{1}{12}(H-L)^2} = \sqrt{\frac{1}{12}(2-1)^2} = \sqrt{\frac{1}{12}} = 0.2887 \text{ days}$$

Since schedules (and scheduling software programs) use discontinuous working days (as opposed to continuous calendar days) to define start and finish dates, the probabilistic finish date will be one *or* two days after the start date, not anywhere within entire range of the distribution. This phenomenon induces changes in the statistics of the finish date of the task and the overall distribution shape and statistics of the schedule. If the duration is treated as a discrete uniform $DU(L,H)$ distribution with two (n=2) discrete days duration, the statistics are:

$$\mu_{DU(1,2)} = \frac{L+H}{2} = \frac{1+2}{2} = 1.5 \text{ workdays (wd)}$$

$$\sigma_{DU(1,2)} = \sqrt{\frac{\left(H-\mu_{DU(1,2)}\right)^2+(L-\mu_{DU(1,2)})^2}{n}} = \sqrt{\frac{(2-1.5)^2+(1-1.5)^2}{2}} = \sqrt{\frac{1}{4}} = 0.5 \text{ wd}$$

Note the mean is unchanged, but the variance increases dramatically because the probability mass is equally distributed at the lower (L) and upper (H) bounds of the distribution. The statistics take a more severe departure when evaluating the distribution in calendar days where one possible finish day may occur on a Friday and another on a Monday, assuming Saturday and Sunday are not workdays. This translates into a distribution with two possible durations in calendar days with the statistics:

$$\mu_{DU(1,4)} = \frac{L+H}{2} = \frac{1+4}{2} = 2.5 \text{ calendar days (cd)}$$

$$\sigma_{DU(1,4)} = \sqrt{\frac{\left(H-\mu_{DU(1,2)}\right)^2+(L-\mu_{DU(1,2)})^2}{n}} = \sqrt{\frac{(4-2.5)^2+(1-2.5)^2}{2}} = \sqrt{\frac{1}{4}} = 1.5 \text{ cd}$$

We must take great care to properly define the appropriate units and respective shapes of durations or else we may be miscalculating the correct moments of the schedule durations, start dates and finish dates. For this reason, probabilistic workdays are defined by continuous distributions, and calendar days are defined by discrete distributions.

3.3.1.1 Converting Calendar Days to Workdays

Scheduling software makes provisions for converting from a number of calendar days to workdays and vice versa. A simple approximation that can be used is:

$$cd = (7/5)wd \pm \varepsilon \text{ where } \varepsilon = 1 \text{ wd} \qquad \textbf{3-8}$$

This conversion provides less than 1% error for date conversions over 10 wd as shown in Figure 3-4. An equally useful approach when using Excel is to compute the finish date (in cd) using the *WORKDAY()* function, which calculates the finish date (in cd) using the start date (in cd) and duration (in wd). The duration in cd (and the appropriate conversion factor from wd to cd) can be calculated by subtracting the finish date (in cd) from the start date (in cd).

3.3.1.2 Expressing Durations and Dates as Random Variables

When probabilistic schedule network tools use continuous distributions to define the probabilistic durations of tasks, they effectively transform the continuous distributions into discrete distributions binned into possible working days. This discretization of continuous distributions scales the standard deviation of the task's duration. The conversion factor shown in 3-8 provides a good approximation of this scaling for standard deviations of durations over 25 wd as shown in Figure 3-4.

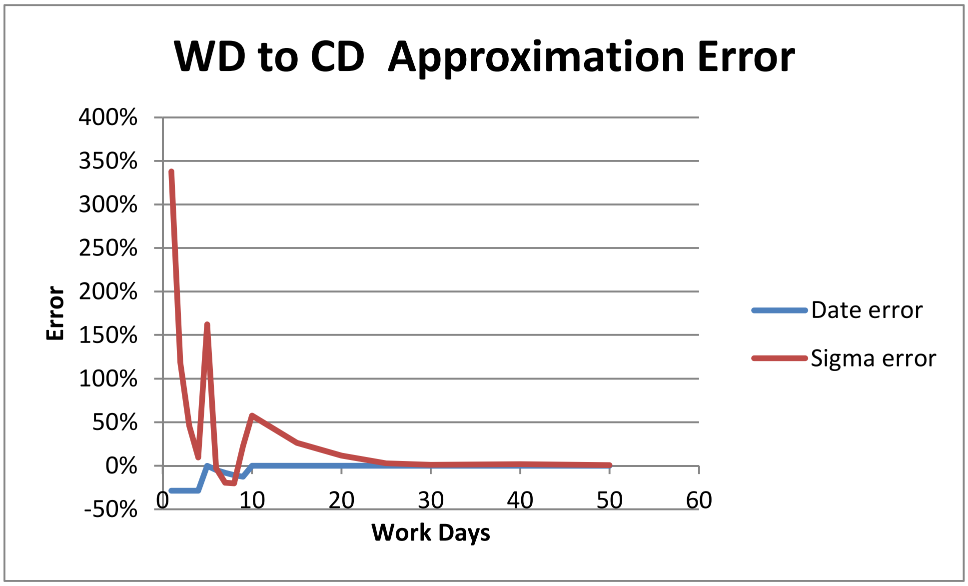

Figure 3-4 Workday-to-Calendar Day Approximation Error

3.3.2 Arrangement of Tasks in a Network

Schedule networks contain the task durations and the arrangement of those tasks with respect to each other. There are four possible arrangements: serial, parallel, tree and feedback (Book S. A., 2011).[16]

[16] Book, S. A., "Schedule Risk Analysis: Why It is Important and How to Do It", 2011 ISPA/SCEA Joint Annual Conference & Training Workshop, Albuquerque NM, 7-10 June 2010.

3.3.2.1 Serial Arrangement

In a serial arrangement, each task is arranged as a predecessor or a successor of another. Figure 3-5 shows a serial arrangement of tasks represented by boxes. The number in each box indicates the duration (number of wd) allocated to the individual tasks. The serial network's critical path passes through all of the boxes, and its duration is the sum of the durations of the individual activities in the serial network. The critical path, in this case, has a total duration equal to 32 wd.

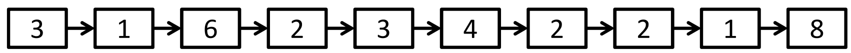

Figure 3-5 Serial Network (Book S. A., 2011)

3.3.2.2 Parallel Arrangement

In a parallel arrangement, two activities are "parallel" if neither is a predecessor or a successor of the other. The critical path passes through those boxes whose combined duration is the longest possible through the network, not the sum of the durations of all of the individual tasks in the network.[17] In Figure 3-6, the series of tasks on the top (the critical path) is outlined in solid lines and have a total duration of 32 wd; the series of tasks at the bottom is outlined in dashed lines and has a total duration of 27 wd.

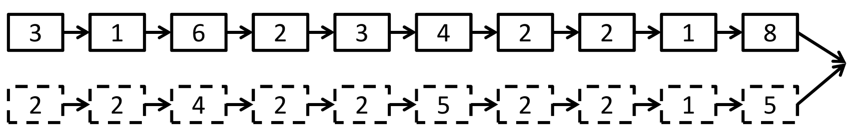

Figure 3-6 Parallel Network (Book S. A., 2011)

3.3.2.3 Tree Structure

A tree structure is a mixture of serial and parallel activities in a schedule network. In Figure 3-7, the numbers in boxes indicate number of workdays allocated to the task represented by each box. The critical path passes through those boxes whose combined duration is the longest possible through the network, not the sum of the durations of all of the individual tasks in the network. The critical path, consisting of boxes outlined in solid lines, has a total duration = 25 wd. The sequences of boxes outlined in dotted black lines have "slack time" of 3 wd, 8 wd, 21 wd, 5 wd and 1 wd, respectively.

[17] The fundamental reason why "Earned Schedule" is an incorrect approach for estimating the expected duration of a program with parallel paths is that the total schedule duration is not equal to the sum of the individual task durations.

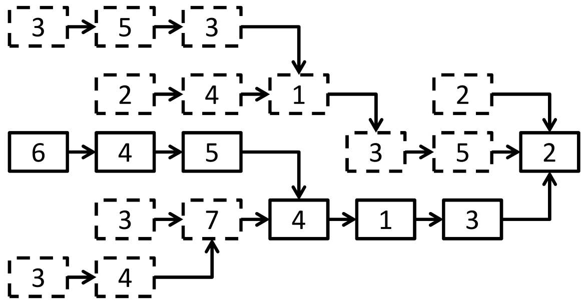

Figure 3-7 Tree-Structured Network (Book S. A., 2011)

The critical path in this case is defined by the maximum of the path durations of each "branch" or path in the tree structure. This is a fundamental difference between schedule-analysis software and cost-analysis software. The work breakdown structure is a "linear" list, and program cost is calculated by adding together the costs of all items on that list. The schedule network (unless it is entirely serial) is not linear, and therefore program duration cannot be calculated by adding together the durations of all activities in the network.

3.3.2.4 Merging Tasks

When parallel branches or tasks in a tree structure merge, the start date of their successor task is driven by the maximum of the end dates of the merging predecessor tasks. The mathematical problem to be solved when dealing with probabilistic schedule analysis (i.e., probabilistic start dates, end dates and durations) where tasks merge is the calculation of the PDF of the maximum, $max(f_X(x))$, of the PDFs of merging tasks (Covert, Using Method of Moments in Schedule Risk Analysis, 2011). This is the source of a phenomenon called "merge bias" which was first discovered in the early 1960s (MacCrimmon & Ryavec, 1962), (Archibald & Villoria, 1967) when a statistical approach was applied to schedule network analysis.[18, 19]

3.3.2.5 Feedback Loop

A feedback loop uses a series of feedback paths to define repeated paths such as repeated testing due to test failures and subsequent fixes. In Figure 3-8, the numbers in boxes indicate the number of wd allocated to the task represented by each box. The critical path passes through those boxes whose combined duration is the longest possible through the network. If "feedback" is not exercised, the critical path, consisting of the boxes outlined in solid lines, has a total duration = 19 wd. If "feedback" is exercised once, all boxes lie on the critical path, which then has total duration = 44 wd.

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[18] MacCrimmon, K. R., & Ryavec, C. A. (1962). An Analytical Study of the Pert Assumptions. Santa Monica, CA: RAND.
[19] Archibald, R. D., & Villoria, R. L. (1967). Network-Based Management Systems (PERT/CPM). New York: John Wiley & Sons.

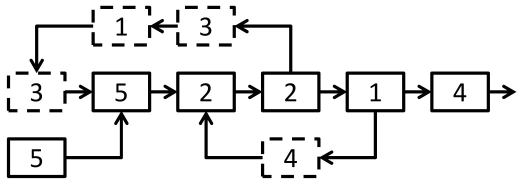


Figure 3-8 Feedback Loop (Book S. A., 2011)

3.3.2.6 Probabilistic Branching

The feedback loop is difficult (and sometimes impossible) to model using commercially available scheduling software, and is often modeled using probabilistic branching techniques. These techniques insert a series of tasks in a schedule network with a set of enabling "switches" based on the probability that these additional or repeated tasks will occur. In Figure 3-9 , the probabilistic switches are indicated by circles (nodes) containing "p", representing the probability of the path being exercised.

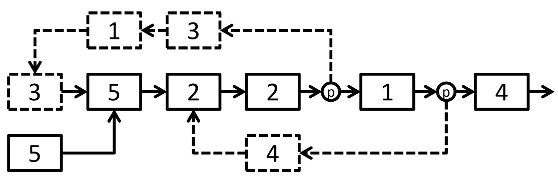


Figure 3-9 Feedback Loop with Probabilistic Decisions

Written in a non-recursive form, the additional, repeated tasks look like those shown in Figure 3-9.

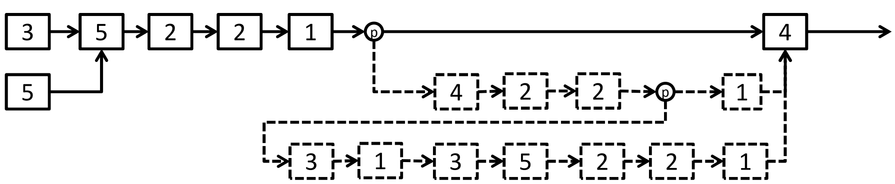


Figure 3-10 Feedback Loop with Probabilistic Branching

Probabilistic branching requires us to know how to add probability-weighted schedule duration (a random variable) to a particular path's duration (another random variable) (Covert, Using Method of Moments in Schedule Risk Analysis, 2011).

3.3.3 The Critical Path

The criticality index (CI) is the probability a particular task's path will be on the critical path, or the probability one path will have a longer duration than the others. Where three parallel paths (A, B and C) with probabilistic end dates merge, there are three potential critical paths, each with its own CI, defined as:

$$CI_A = P(A > max(B, C))$$

$$CI_B = P(B > max(A, C))$$
$$CI_C = P(C > max(B, A))$$

Generally, we can state the CI of path X (CI_X) to be

$$CI_{X_i} = P(X_i > max(X_{j \neq i})) \quad \textbf{3-9}$$

Using the notation for the maximum of distributions to be X, then the probability that the end date of path A is greater than the maximum of paths B and C, P(A>X), which is the same as P(X<A), and therefore P(X-A<0). We will need to know how to subtract two correlated random variables (the probabilistic durations of the individual paths in the network) to compute the CI (Covert, Using Method of Moments in Schedule Risk Analysis, 2011).[20]

3.4 Mathematics of Estimates

In Sections 3.2 and 3.3, we discussed mathematical problems to be solved when using a variety of cost and scheduling estimating methods. The mathematical operations applied to random variables in which we are most interested are (Figure 3-11): addition and subtraction, multiplication and division, correlation between random variables, minimum and maximum, linear and nonlinear transformations, and discrete risks and probabilistic branching. These operations between PDFs result in new PDFs with moments of their own, which we will use in the analysis. What we have not discussed yet is the subject of correlation of random variables, which affects all of these operations.

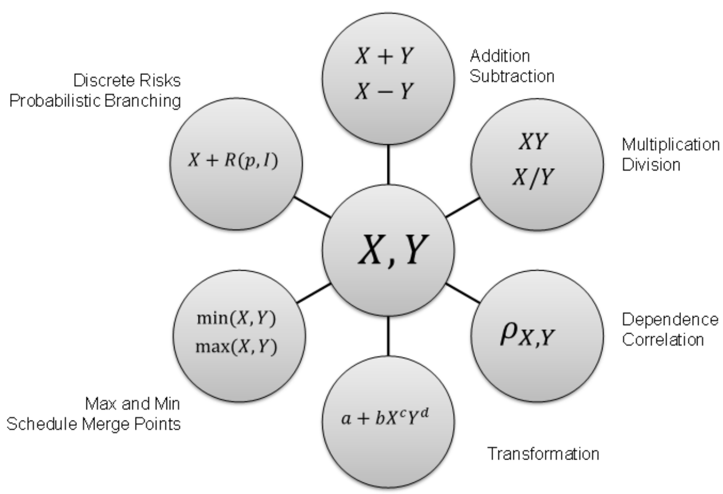


Figure 3-11 Mathematics of Random Variables

[20] Covert, R. P. (2011). Using Method of Moments in Schedule Risk Analysis. Bethesda, MD: IPM.

3.4.1 Correlation between Random Variables

When performing operations on random variables we must have knowledge of how they behave with respect to each other, or covary. Correlation is a statistical measure of association between two random variables and is specified by a correlation coefficient ($\rho_{i,j}$). It measures how strongly the random variables are related, or change, with each other. If two random variables tend to move up or down together, then they are said to be positively correlated. If they tend to move in opposite directions, they are said to be negatively correlated. The most common statistic for measuring association is the Pearson (linear) correlation coefficient, ρ. Another is the Spearman (rank) correlation coefficient, ρ_S, which is used in statistical simulation tools such as Crystal Ball and @Risk. These two definitions of correlation are different, and should not be confused to mean the same thing. Garvey (1999) pointed out that simulations relying on rank correlation do not correctly model the covariance of random variables.[21]

Pearson product-moment linear correlation, $\rho(X,Y)$, measures the extent of linearity of a relationship between two random variables. It plays an explicit, well-defined role in establishing the sigma value (as well as the range) of the total-cost distribution as described by Book (1994). For example:

- $\rho(X,Y) = \pm 1$ if and only if (iff) *X* and *Y* are linearly related, i.e., the least-squares linear relationship between *X* and *Y* allows us to predict *Y* precisely, given *X*
- $\rho^2(X,Y)$ = proportion of variation in *Y* that can be explained on the basis of a least-squares linear relationship between *X* and *Y*
- $\rho(X,Y) = 0$ iff the least-squares linear relationship between *X* and *Y* provides no ability to predict *Y*, given *X*

The second type of correlation, called Spearman rank correlation, $\rho_S(X,Y)$, measures the extent of monotonicity of a relationship between two random variables. Since it does not appear explicitly in the formulae for any of the mathematical operations for which we are concerned, its impact on sigma is not known.

- $\rho_S(X,Y) = +1$ iff the largest value of *X* corresponds to the largest value of *Y* , the second largest, ... , etc.
- $\rho_S(X,Y) = -1$ iff the largest value of *X* corresponds to the smallest value of *Y*, etc.
- $\rho_S(X,Y) = 0$ iff the rank of a particular *X* among all *X* values. In this case it provides no ability to predict the rank of the corresponding *Y* among all *X* values

[21] Garvey, P. R. (1999). Do Not Use Rank Correlation in Cost Risk Analysis. 32nd DOD Cost Analysis Symposium.

Linear and rank correlations are different for different sets of pairwise data. As an example, Figure 3-12 shows the linear and rank correlation coefficients for different plots of x and y variables.[22]

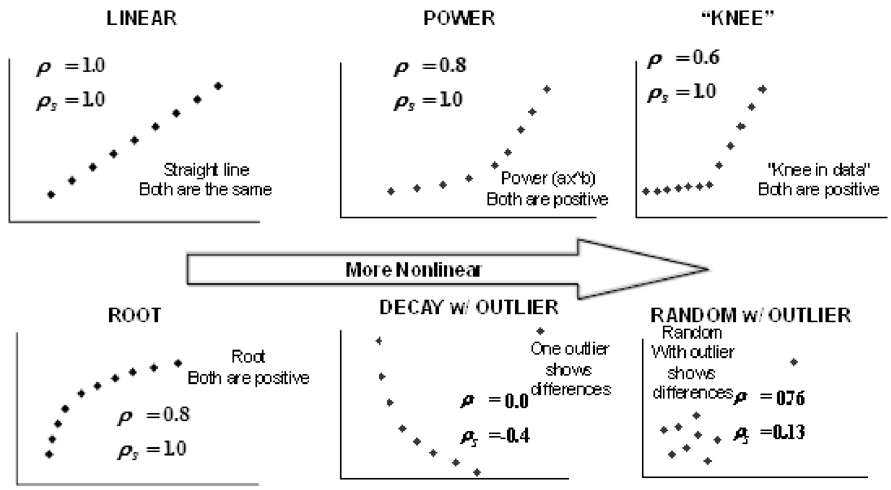


Figure 3-12 Linear vs. Rank Correlation

We discuss these two types of correlation because: 1) Pearson product-moment correlation is an essential element used to find the distributions formed by mathematical operations on random variables, 2) Spearman correlation is used nearly exclusively in statistical simulations and does not define covariance, and 3) we need to know the difference between them if we are interested in comparing analytical results to those produced by statistical simulations.

3.4.2 Calculating Correlation Coefficients

The correlation coefficient between lists of values of random variables, such as the multiplicative (or additive) error terms of CERs, can be calculated quite easily. Previous papers by the author (2001), (2002), (2006) have demonstrated this application. [23, 24, 25] The Pearson product-moment correlation between discrete values such as pair-wise CER residuals is calculated using Equation 3-10.

$$\rho_{X,Y} = \frac{\sum(X_i - \mu_X)(Y_i - \mu_Y)}{\sqrt{\sum(X_i - \mu_X)^2 \sum(Y_i - \mu_Y)^2}} \qquad \textbf{3-10}$$

[22] Covert. R. P. (2011). Using Method of Moments in Schedule Risk Analysis. Bethesda, MD: IPM.
[23] Covert, R. P. (2001). Correlation Coefficients in the Unmanned Space Vehicle Cost Model Version 7 (USCM 7) Database. 3rd Joint ISPA/SCEA International Conference. Tyson's Corner, VA.
[24] Covert, R. P. (2002). Comparison of Spacecraft Cost Model Correlation Coefficients. SCEA National Conference. Scottsdale, AZ.
[25] Covert, R. P. (2006). Correlations in Cost Risk Analysis. 2006 Annual SCEA Conference. Tysons Corner, VA.

where X and Y are CER residual pairs,
X_i and Y_i are individual program residual data, and
μ_X and μ_Y are the means of the residuals respectively.

If the two variables exactly follow a linear relationship (with no scatter), then the correlation coefficient $\rho_{X,Y}$ = +1 or -1. Similarly, if there is no correlation between X and Y, then the numerator should be zero, and $\rho_{X,Y} = 0$.

3.4.3 Correlation, Dependence and Independence

In the process of researching the analytic method presented in this paper, we found correlation can be induced between two vectors of sampled, uncorrelated variables X and Y when one, the other, or both are transformed through a non-linear equation (i.e., a CER) form such as $y = aX^b$, or a triad type of CER, $y = a+bX^c$.

Consider the two uncorrelated random variables U and V shown in Table 3-2. We will introduce a linear transformation, $W = 2 + 3U$, and two exponential transformations, $X = U^2$ and $Y = V^2$. A linear transformation does not change the fundamental correlation, as seen in the correlation coefficients $\rho_{U,W}$ and $\rho_{V,W}$ (Table 3-3). Small amounts of correlation are induced by the exponentiation of the uncorrelated random variables U and V as seen in $\rho_{U,Y} = -0.0088$, and $\rho_{V,X} = 0.1925$. Variables correlated with their squares show a decrease in their correlation from 1.0 as seen in $\rho_{U,X} = 0.9811$ and $\rho_{V,Y} = 0.9990$.

Table 3-2 Transformed Random Variable Samples

U	V	W=2+3U	X=U^2	Y=V^2
1	4.2	4	1	17.64
2	2.1	6	4	4.41
3	1.8	8	9	3.24
4	2.2	10	16	4.84
5	4.15	12	25	17.2225

Table 3-3 Correlations between Transformed Random Variables

	U	V	W	X	Y
U	1.0000	0.0000	1.0000	0.9811	-0.0088
V	0.0000	1.0000	0.0000	0.1924	0.9990
W	1.0000	0.0000	1.0000	0.9811	-0.0088
X	0.9811	0.1924	0.9811	1.0000	0.1828
Y	-0.0088	0.9990	-0.0088	0.1828	1.0000

This demonstration shows that while any pair of sampled vectors of random numbers may themselves be uncorrelated, their exponentiated values are not (i.e., $\rho_{U,V} \neq \rho_{U^2,V^2}$). While we may believe we have two sample vectors of independent random variables, we probably do not. True statistical independence is a high standard of independence between random variables and is difficult to achieve – particularly through statistical sampling. A less stringent type of independence is "expectation independence", in which the variables remain uncorrelated (i.e., $\rho_{U,V} = \rho_{U^k,V^k} = 0$) for any higher order of expectation operations. "Uncorrelated" is the least stringent standard, and as our demonstration shows, correlation can be induced through exponentiation of the random variables.

Another way RVs can be correlated is through the structure of the mathematical problem (i.e., the functional relationship to each other directly through one equation or indirectly through more than one equation), whether that structure is a cost estimate or a schedule network. In a cost estimate, two CERs can be correlated through sharing a common cost driver or where one CER drives another CER, such as a cost-on-cost factor. Garvey[26] (2000) provides an analytic method of determining $\rho_{X,Y}$ when X and Y are random variables representing the estimates from errorless CERs. In a schedule network, two finish dates may have uncorrelated durations of their predecessor tasks, but will still be correlated to each other by sharing a common predecessor. We are interested in calculating functional correlation out of necessity when using analytic methods of uncertainty analysis.

[26] Garvey, P. R. (2000). Probability Methods for Cost Uncertainty Analysis: A Systems Engineering Perspective. New York, NY: Marcel Dekker.

4 Probability Tools

When we use a cost model to perform a cost risk analysis, we need to know the uncertainty of the individual cost estimates, their statistical dependencies, and how to calculate their sums. We can employ statistical modeling techniques such as statistical simulation or statistical analysis to find these uncertainties and their properties. Although the goal is the same, these techniques differ, which we will discuss in more detail.

4.1 Statistical Simulation

Statistical simulation is a numerical experiment designed to provide statistical information about the properties of a model driven by random variables. It is often used in cost and schedule risk analysis to model the complex interaction of the transformations and summations involved with correlated random variables.

The statistical simulation process follows these steps:

1) Define numerical experiment (spreadsheet, schedule network, etc.)
2) Define PDFs for each random variable
3) Define correlation coefficients for random variables
4) Determine the number of experimental trials
5) For each trial:
    a. Draw correlated random variable(s) from defined PDF(s)
        i. Sample uniform distributions, $U(1,0)$
        ii. Transform each $U(1,0)$ to the desired PDF based on an inverse transformation of the cumulative density function (CDF), denoted as CDF^{-1}.
        iii. Correlate the set of PDFs
    b. Compute the experimental result(s)
    c. Save the experimental result(s)
6) At the end of the simulation, determine the statistics from the experimental results

4.1.1 Sampling Techniques

Statistical simulation tools use one or more of the following sampling techniques:

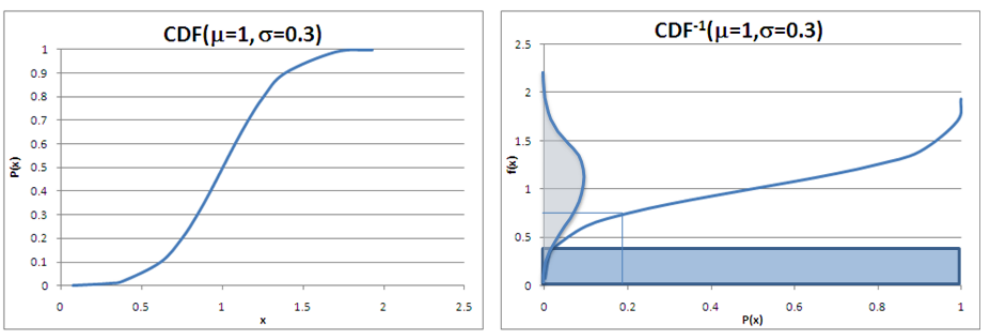
- Bootstrap sampling: Re-sampling with replacement from sample data numerous times in order to generate an empirical distribution of a statistic
- Monte Carlo sampling: New sample points are generated without taking into account the previously generated sample points
- Latin Hypercube sampling: Each variable is divided into m equally probable divisions and sampling is done without replacement for each set of m trials
- Orthogonal sampling: This adds the requirement that the entire sample space must be sampled evenly

The most commonly-used statistical simulations use Monte Carlo or Latin Hypercube sampling of correlated random variables. The reasonableness of the simulation results depends on the reasonableness of the user inputs, correct modeling of PDFs for all random variables, and the correct specification of the correlation between these PDFs (even if it is assumed to be 0). The accuracy of the simulation is highly dependent on the simulation's ability to draw uniformly-distributed random variables $U(1,0)$ in step 5.a.i and to correlate them correctly in step 5.a.iii.

4.1.1.1 Generating PDFs from Random Number Generators

A random number generator, such as the Excel *RAND( )* statement, produces a uniformly-distributed pseudo-random number between 0 and 1 ($0 \leq U(0,1), \leq 1$). We know that the range of the CDF, $F_X(x)$, for any random number is the same (i.e., $0 \leq F_X(x) \leq 1$). Based on that knowledge, the uniform draw can be transformed by the inverse of the CDF, the CDF^{-1}, to get the desired probability distribution, $f_X(x)$ as shown in Figure 4-1. The Excel statements are fairly simple to use for this purpose, as we will demonstrate.

We can generate different PDFs using Excel to demonstrate how statistical simulations generate differently-distributed random numbers. First, we will generate a pseudo-random number based on a uniform distribution $U(0,1)$, then transform it into the desired PDF using the inverse CDF (i.e., CDF^{-1}) using simple Excel functions.



Note: In the graph on the left, the cumulative probability, P(x), is the vertical axis, and in the graph on the right, P(x) is the horizontal axis.

Figure 4-1 Simulating a Lognormal Distribution

In our example, 1000 uniformly-distributed numbers over the interval [0,1] were generated using the Excel *RAND( )* function. Figure 4-2 shows the histogram of the 1000 uniform draws, which is a representation of $U(0,1)$.

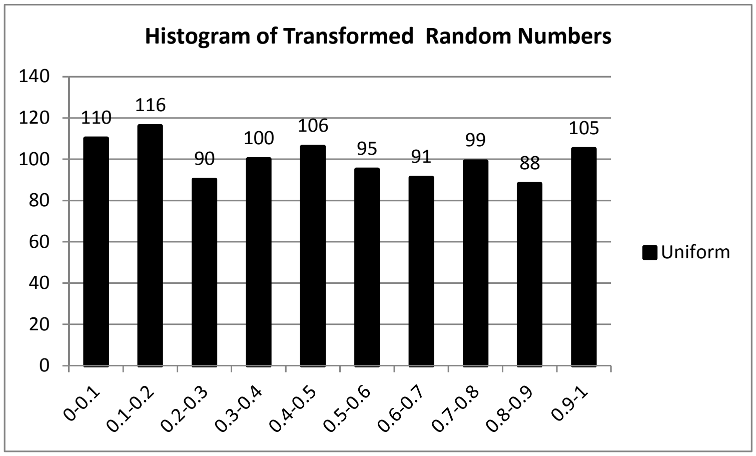


Figure 4-2 Simulated Uniform Distribution

The moments of the pseudo-random uniform distribution formed by the 1000 samples, the vector $\boldsymbol{Y}$, can be easily calculated using the following Excel statistical functions:

- $\mu = AVERAGE(\boldsymbol{Y})$
- $\sigma = STDEV(\boldsymbol{Y})$
- $\vartheta = SKEW(\boldsymbol{Y})$
- $\kappa = KURT(\boldsymbol{Y})$

Note the kurtosis calculated by the Excel function is excess kurtosis. The moments of the uniform samples and their exact values based on the defined uniform distribution are shown in Table 4-1.

Table 4-1 Moments of the Simulated Uniform Distribution

Moment	Simulated	Exact
μ	0.488	0.500
σ	0.292	0.083
ϑ	0.053	0.000
κ	-1.222	-1.200

Based on the moment statistics of the uniform distribution, it is slightly biased low (based on the mean), somewhat unevenly distributed (based on the standard deviation), right-skewed (based on the positive skewness), and platykurtic (based on the excess kurtosis).

A normal distribution $N(1000{,}300)$ can be generated by transforming $U(0{,}1)$ using the inverse CDF of a normal distribution. The transform function (i.e., the inverse CDF of a normal distribution) used in this example is *NORMINV* (x, μ, σ),[27] where x is the draw from $U(0,1)$, μ = 1000, and σ = 300. Figure 4-3 shows the histogram of the normal PDF formed by this procedure, and Table 4-2 shows the moments of the simulated and exact values expected.

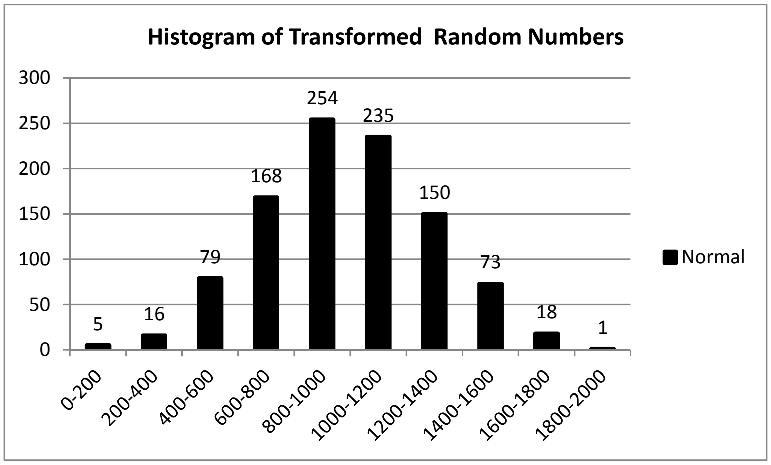


Figure 4-3 Simulated Normal Distribution

Table 4-2 Moments of the Simulated Normal Distribution

Moment	Simulated	Exact
μ	987.7155	1000
σ	303.4236	300
ϑ	0.001349	0
κ	-0.12993	0

Likewise, a lognormal distribution $L(1000{,}300)$ can be generated by transforming $U(0,1)$ using the inverse CDF of a lognormal distribution. The transform function used in this example is *LOGINV(x,P,Q)*.[28] Before we can use the inverse lognormal transformation, we must find *P* and *Q*, which are the log-transformed mean and sigma of the lognormal distribution. The log-transformed mean, $P = \frac{1}{2} ln\left(\frac{\mu^4}{\mu^2+\sigma^2}\right) = 6.8647$, and the log-transformed sigma, $Q = \sqrt{ln\left(1 + \frac{\sigma^2}{\mu^2}\right)} = 0.2936$.

[27] NORMINV( ) is an Excel 2007 function, and NORM.INV( ) is an Excel 2010 function.
[28] LOGINV( ) is an Excel 2007 function and LOGNORM.INV( ) is an Excel 2010 function.

Figure 4-4 shows the histogram of the lognormal PDF formed by this procedure, and Table 4-3 provides the moments of the simulated and exact values expected.

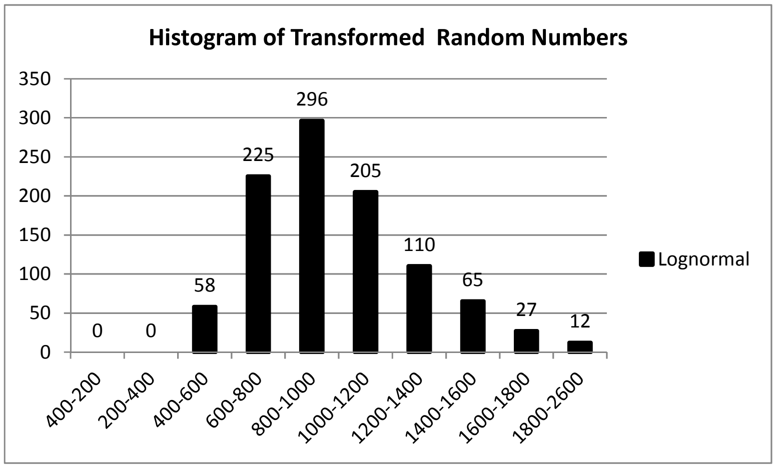


Figure 4-4 Simulated Lognormal Distribution

Table 4-3 Moments of the Simulated Lognormal Distribution

Moment	Simulated	Exact
μ	988.989	1000
σ	299.102	300
ϑ	0.855934	0.927
κ	1.094075	1.566

4.1.2 Correlating Random Numbers

Much literature in the statistics community exists regarding generating correlated random numbers for use in statistical simulation, but few families of joint PDFs specified in terms of their Pearson product-moment correlation exist. Among ones that do exist are correlated joint normal, joint normal-lognormal and joint lognormal distributions discussed in Garvey (2000).[29] Other families of joint distributions are formed through the use of copulas – a transformation technique used to create joint probability distribution.

4.1.3 Timing of Discovery of Correlation Methods

The timing of the discovery of methods of generating correlated random numbers was an influence on which commercially-available risk analysis tools use Pearson (product moment) correlation vs. Spearman (rank) correlation. Commercial tools developed in the early-1980s (i.e., @Risk and Crystal Ball) use a method of generating rank correlated

[29] Garvey, P. R. (2000). Probability Methods for Cost Uncertainty Analysis: A Systems Engineering Perspective. New York, NY: Marcel Dekker.

random numbers based on a published paper (Iman & Conover, 1982)[30]. In the late-1990s, a new algorithm (Lurie & Goldberg, 1998)[31, 32] was published that provided a method of generating Pearson-correlated random numbers. Many of the commercially available statistical simulation tools were developed before the Lurie-Goldberg paper, so they rely on Spearman rank correlation. However, these are limitations of using rank correlation when performing cost risk analysis as noted in Garvey's paper[33] (1999). Only since 1998 have tools such as Risk+ for Microsoft Project been programmed with the method presented by Lurie and Goldberg.

4.1.4 Benefits and Drawbacks of Statistical Simulation Techniques

Statistical simulation has its benefits and drawbacks. Among its benefits are 1) its ability to provide the statistics of a simulated PDF formed by complex mathematical modeling of random variables and 2) its relative ease of use. Quite often, statistical simulation obtains very close results to and is easier to use than statistical analysis. However, statistical simulation does have its drawbacks – particularly due to its 1) inability to sample uniformly, 2) (in)ability to correlate two distributions exactly using Pearson product-moment correlation coefficients, 3) difficulty of correlating large numbers of random variables, and 4) inability to provide reasonable results when the number of simulation trials is too small to account for single or combinations of low-probability events. The last error is further exaggerated when multiplying highly-skewed random variables (e.g., the product of two lognormal PDFs) and when performing discrete risk analysis. In these instances, high-impact, low-probability-of-occurrence events are difficult for simulations to adequately sample in order to produce reasonable facsimiles of the exact results.

One way to check the reasonableness of the results of a statistical simulation is to: 1) "dump" a list of the results of the correlated random variables being modeled, 2) calculate the resulting statistics (e.g., Pearson correlation coefficient between the variables), and 3) find the fit statistics of the distributions being modeled. By performing a dump of the simulated variables, an analyst will be able to ensure the simulation has created a reasonable facsimile of the desired input distributions and output distributions (or the calculation of the Pearson correlation between the correlated random variables) and that they are close to that specified. Any statistical simulation tool that does not provide the ability to examine a dump of the trials should be avoided.

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[30] Iman, R.L. and Conover, W.J., "A Distribution-free Approach to Inducing Rank Correlation among Input Variables," Communications in Statistics - Simulation, Computation, Vol. 11, No. 3(1982), pages 311-334.

[31] Lurie, P.M.; Goldberg, M.S., "A Method for Simulating Correlated Random Variables from Partially Specified Distributions," Management Science, Vol. 44, No. 2, February 1998, pages 203-218.

[32] Related briefing: "Simulating Correlated Random Variables," 32nd DOD Cost Analysis Symposium, 2-5 February 1999.

[33] Garvey, P.R., "Do Not Use Rank Correlation in Cost Risk Analysis," 32nd DOD Cost Analysis Symposium, 2-5 February 1999.

4.2 Statistical Analysis

Unlike simulation, statistical analysis relies on the exact calculation of moments of the PDF. We will use moments as the basis of the analytical technique proposed in this report.

4.2.1 Moments

Moments are important measures of the properties of random variables, and they come in many varieties. The three we have discussed earlier and with which we are most concerned are raw moments, central moments and standardized moments.

4.2.2 Method of Moments

Method of Moments (MOM) is a relatively easy-to-use, analytical technique used to calculate the moments of probability distributions. The MOM technique relies on exact statistical calculations of moments to derive the statistics of probability distributions such as WBS element cost estimates or schedule durations. With the widespread use of statistical simulation tools by cost and schedule analysts, MOM has become a forgotten "art". One of the surviving MOM techniques is the Formal Risk Assessment of System Cost Estimates (FRISK) method (Young, 1992).[34]

4.2.2.1 FRISK

FRISK is a MOM approach used to calculate the μ and σ^2 of the PDF of total cost formed by the statistical summation of PDFs of subordinate cost elements.

The steps used in the FRISK method are:

1. Define numerical experiment; in this case, the summation structure of a WBS
2. Define triangular PDFs, $T(L_i, M_i, H_i)$ for each cost, X_i , or random variable to be statistically summed, by specifying the low (L_i), most likely (M_i) and high (H_i) values
3. Calculate the μ_i and ${\sigma_i}^2$ for each $T(L_i, M_i, H_i)$ using Equations 4-1 and 4-2

$$\mu_i = (L_i + M_i + H_i)/3 \qquad \textbf{4-1}$$

$${\sigma_i}^2 = ({L_i}^2 + {M_i}^2 + {H_i}^2 - L_i M_i - L_i H_i - M_i H_i)/18 \qquad \textbf{4-2}$$

4. Sum the n means to calculate the mean of the sum of the PDFs using Equation 4-3

$$\mu_{Tot} = \sum_{i=1}^{n} \mu_i \qquad \textbf{4-3}$$

5. Define correlation coefficients, $\rho_{i,j}$, for each pair of PDFs
6. Calculate the total variance of the sum of the PDFs using Equation 4-4

$${\sigma_{Tot}}^2 = \sum_{i=1}^{n} {\sigma_i}^2 + \sum_{i>j} \sum_{j=1}^{n-1} \rho_{i,j} \sigma_i \sigma_j \qquad \textbf{4-4}$$

7. Assume the PDF of the total cost is a lognormal distribution, $L(P, Q)$

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[34] Young, P. H. (1992). FRISK - Formal Risk Assessment of System Cost. Aerospace Design Conference. Irvine, CA: AIAA.

8. Calculate the lognormal parameters P and Q using Equations 4-5 and 4-6.

$$P = \frac{1}{2} ln\left(\frac{\mu^4}{\mu^2+\sigma^2}\right) \quad \textbf{4-5}$$

$$Q = \sqrt{ln\left(1+\frac{\sigma^2}{\mu^2}\right)} \quad \textbf{4-6}$$

9. Determine the percentile statistics $L(P,Q)_Z$ using the inverse CDF tables or the *LOGINV* function in Excel.

The outputs from an example FRISK calculation are shown in Figure 4-5.

Percentile	Value
10%	517.99
20%	542.63
30%	561.12
40%	577.42
50%	593.08
60%	609.17
70%	626.86
80%	648.23
90%	679.06

Statistics	Value
Mean	596.40
Median	593.08
Mode	586.50
Standard Deviation	63.18

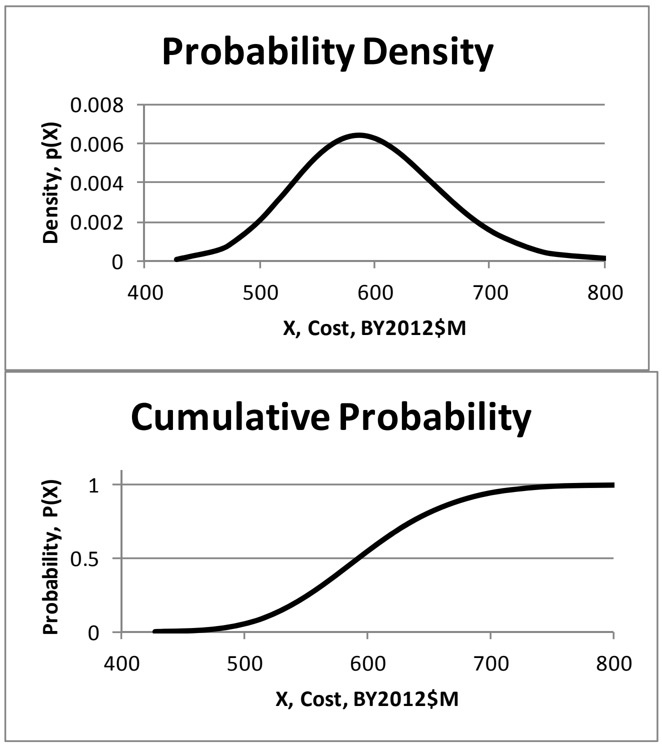

Figure 4-5 Example FRISK Output

FRISK is even more efficient when programmed as an Excel spreadsheet. The means and standard deviations of triangular distribution inputs in step 3 can be calculated using *AVERAGE(L,M,H)* and *STDEVP(L,M,H)/2*, respectively. When the series of means and variances to be statistically summed appears in contiguous cells (rows or columns), the following Excel functions can be used:

1. *SUM(range)*, where range is the series of means
2. *SQRT(MMULT(TRANSPOSE(*$\vec{\sigma}$*),MMULT(**R**,*$\vec{\sigma}$*)))*, where $\vec{\sigma}$ is the range of the vector of σ_i in columnar form and ***R*** is the *nxn* correlation matrix. This function must be entered by pressing *<CTRL> <SHIFT> <ENTER>*. An example of the correlation matrix is shown in Figure 4-6.

$$\begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}$$

Figure 4-6 Example Correlation Matrix

When all μ_i and σ_i used in the statistical summation are not in contiguous cells, we can re-create a set of contiguous cells elsewhere in the spreadsheet (or through an Excel macro) to allow the use of the Excel functions (1 and 2) above.

Let us perform an example FRISK rollup calculation using a set of errorless estimating relationships from Book (1994).[35] Assume we have modeled the cost estimates of the WBS elements with triangular distributions as shown in Table 4-4. The parameters of the triangular distributions are the outputs of a CER using Low, Most Likely and High cost drivers.

Table 4-4 Example FRISK Rollup Inputs (Costs in $K)

WBS Element, i	L_i	M_i	H_i
Antenna	191	380	1151
Electronics	96	192	582
Platform	33	76	143
Facilities	9	18	27
Power Distribution	77	154	465
Computers	30	58	86
Environmental Control	11	22	66
Communications	58	120	182
Software	120	230	691
TOTAL	625	1250	3393

Note the naïve sum of the most likely costs, M_i, is $1250K.

The first WBS element, the Antenna WBS element CER, is defined by a triangular distribution, $T(191{,}380{,}1151)$. The mean of a triangular distribution from Equation 4-1 is

$$\mu_1 = (L_1 + M_1 + H_1)/3 = \frac{191+380+1151}{3} = \$574K \quad \textbf{4-7}$$

and the standard deviation of the Antenna WBS cost using Equation 4-2 is

[35] Book, S. A. (1994). Do Not Sum 'Most Likely' Cost Estimates. 1994 NASA Cost Estimating Symposium. Houston, TX.

$$\sigma_{f(x)_1} = \sqrt{{\sigma_{f(x)_1}}^2} = \sqrt{\frac{[191^2+380^2+1151^2-(191)(380)-(191)(1151)-(380)(1151)]}{18}} = \$207.62\text{K} \qquad \textbf{4-8}$$

Repeating this procedure for all of the WBS elements in Table 4-4 allows us to calculate the moments ($\mu_{f(x)_i}$ and $\sigma_{f(x)_i}$) for all WBS elements as shown in Table 4-5. The mean of the total is calculated using Equation 4-3. To calculate the total cost sigma, we need to specify a correlation matrix. For this example, we use the matrix shown in Figure 4-6. To calculate the standard deviation of the total, we use the matrix form of Equation 4-4 to obtain the results shown in Table 4-5.

The mean cost is $1756K, which is significantly larger than the naïve sum of the most likely costs, which is $1250K (Book, 1994).[36]

Table 4-5 Example FRISK Rollup (costs in $K)

WBS Element, i	Estimate, $f(x)_i$	$\mu_{f(x)_i}$	$\sigma_{f(x)_i}$
Antenna	T(191,380,1151)	574	207.62
Electronics	T(96,192,582)	290	105.08
Platform	T(33,76,143)	84	22.63
Facilities	T(9,18,27)	18	3.67
Power Distribution	T(77,154,465)	232	83.86
Computers	T(30,58,86)	58	11.43
Environmental Control	T(11,22,66)	33	11.88
Communications	T(58,120,182)	120	25.31
Software	T(120,230,691)	347	123.68
TOTAL (Not necessarily the sum)		1756	364.93

We quantify the percentile value of the sum of the most likely costs by forming a CDF. If we assume the total cost of our estimate is lognormally distributed, we can compute the lognormal distribution parameters ($P = 7.4497$ and $Q = 0.2056$) using Equations 4-5 and 4-6.

A quick calculation using the lognormal distribution functions in Excel tells us the percentile of the naïve sum of most likely costs. The equation and results are:

LOGNORM.DIST (1250, P, Q, TRUE) =0.060553=6.0553%

This is why we model estimates probabilistically. It would be very difficult to defend an estimate at the 6th percentile and unwise to want it in the first place!

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[36] Ibid.

Using the inverse of the lognormal distribution, we find the cost value at any probability level on the CDF. This is a very simple way of quickly forming CDFs such as the one shown in Figure 4-7.

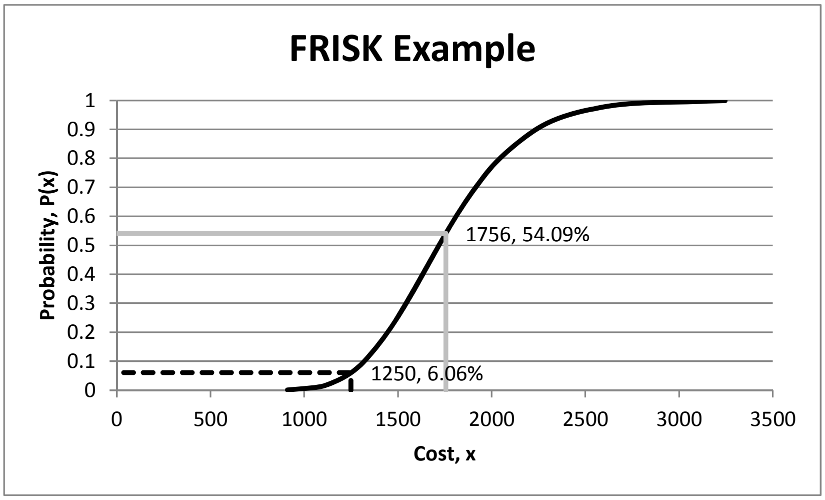

Figure 4-7 FRISK Example CDF

4.2.2.2 Enhancements to FRISK

FRISK is an elegant way to model the simple statistical summation of a cost estimate. However, to be fully effective as a tool to exactly and efficiently analyze a cost estimate, we need to be able to accommodate 1) statistical summation of non-adjacent cells; 2) inputs that are non-triangularly distributed, such as normal or lognormal distributions; 3) modeling CER cost-driver uncertainties, 4) transformation of cost-driver PDFs by a CER, 5) modeling the additive or multiplicative error of the CER, and 6) multi-level summations as in the case of a complex WBS. Fortunately, solutions to these issues are available from the literature (Covert R. P., 2006).[37]

4.3 MOM Operations and Analytic Method Description

This section describes the mathematical treatment of these operations on random variables and provides methods of calculating the moments.

4.3.1 Addition and Subtraction of Random Variables

The simplest mathematical operation with which we will be concerned is the statistical summation and subtraction of random variables.

As we discussed in Section 3.2.1 the WBS defines the summation of individual WBS elements to higher hierarchical levels. Similarly, in Section 3.3.2.1, the serial arrangement of schedule tasks allows us to statistically sum their durations. Both mathematical

[37] Covert, R. P. (2006). Correlations in Cost Risk Analysis. 2006 Annual SCEA Conference. Tysons Corner, VA.

problems are treated with the same statistical summation technique. Let X_i be the cost (or duration) of an individual WBS element (or serially arranged set of schedule tasks), and X_T be the sum of individual WBS elements, i. Then the mean of WBS element i is the expected value $E[\cdot]$ of the random variable, X_i.

$$\mu_i = E[X_i] \qquad \textbf{4-9}$$

So the mean of the sum of individual WBS elements is the total mean, μ_T

$$\mu_T = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] \qquad \textbf{4-10}$$

More simply put, the mean of the sum is the sum of the means.

The total variance, ${\sigma_T}^2$, of the sum of the WBS elements is the square of the standard deviation of the total, σ_T.

$${\sigma_T}^2 = Var(X_T) = \sum_{i=1}^{n} {\sigma_i}^2 + \sum_{i \neq j} \sum \rho_{i,j} \sigma_i \sigma_j \qquad \textbf{4-11}$$

In expectation parlance, Equations 4-12 and 4-13 are the expected values of the sum and difference of two random variables.[38]

$$E[X + Y] = E[X] + E[Y] \qquad \textbf{4-12}$$
$$E[X - Y] = E[X] - E[Y] \qquad \textbf{4-13}$$

Equations 4-14 and 4-15 are the variances of the sum and difference of two random variables. Less intuitive is the variance resulting from the difference of two random variables. Equation 4-15 is similar to Equation 4-14 except the covariance term $2Cov(X, Y)$ is subtracted from the sum of the variances of X and Y.

$$Var[X + Y] = Var[X] + Var[Y] + 2Cov(X, Y) \qquad \textbf{4-14}$$
$$Var[X - Y] = Var[X] + Var[Y] - 2Cov(X, Y) \qquad \textbf{4-15}$$

The shape of the distribution formed by the sum and difference of lognormally distributed random variables is discussed in the applied statistics literature (Lo, 2012).[39] It is agreed that the shape of the sum or difference of two correlated lognormal variables are neither normal nor lognormal, but an approximate shape can be derived from the parameters of the distributions.

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[38] When calculating the criticality index (CI) of a schedule task, we must evaluate the integral of the difference of random variables.

[39] Lo, C. F., The Sum and Difference of Two Lognormal Random Variables (May 22, 2012). Available at SSRN: http://ssrn.com/abstract=2064829 or http://dx.doi.org/10.2139/ssrn.2064829

The parameters of interest when subtracting one lognormally distributed PDF from another are: the correlation between the two PDFs, and their respective means and standard deviations (or variances). These parameters not only determine the mean and variance of the PDF formed by their difference but also the skewness and kurtosis of the same. To estimate the shape of the distribution formed by subtracting one RV from another, we use the results of a numerical experiment (i.e., a 100,000-trial statistical simulation).

The numerical experiment uses four PDFs defined as lognormal distributions: $A = L(1,1)$, $B = L(1,0.5)$, $C = L(2,1)$, and $D = L(2,0.5)$. Table 4-6 shows the difference between uncorrelated pairs (i.e., $\rho = 0$) of A, B, C, and D. We show the mean, standard deviation, skewness, kurtosis and shape of the PDF-defined difference in each of the twelve cases.

Table 4-6 Difference of Two Uncorrelated PDFs

Case	Difference	μ	σ	ϑ	κ	Fit Shape
1	$A - B$	0.000	1.1159	2.613	22.771	Logistic
2	$A - C$	-1.000	1.4152	0.772	11.785	Student's t
3	$A - D$	-1.000	1.1151	2.652	22.033	Max Extreme
4	$B - A$	0.000	1.1159	-2.613	22.771	Logistic
5	$B - C$	-1.000	1.1177	-1.022	6.381	Logistic
6	$B - D$	-1.000	0.7070	0.299	4.471	Logistic
7	$C - A$	1.000	1.4152	-0.772	11.785	Student's t
8	$C - B$	1.000	1.1177	1.022	6.381	Lognormal
9	$C - D$	0.000	1.1198	1.099	6.263	Lognormal
10	$D - A$	1.000	1.1151	-2.652	22.033	Weibull
11	$D - B$	1.000	0.7070	-0.299	4.471	Logistic
12	$D - C$	0.000	1.1198	-1.099	6.263	Weibull

A lognormal PDF is defined by its mean and standard deviation, is right skewed, and it is supported over the range of real values $[0, \infty]$. The mean and standard deviation are always positive real numbers, so a lognormal PDF must have a positive mean and positive skewness. Only case 8 in Table 4-6 can be considered an approximation to a true lognormal distribution based on its mean and skewness. Case 5 produces a mirror image of case 8, so it is considered to be a “negative lognormal distribution”.

We can use the knowledge that if the difference of two RVs (i.e., X-Y) produces a negative lognormal distribution, then all of the area of the PDF of X-Y is in the negative axis. Since this is true, Y-X is a lognormal distribution, and all of its area lies on the positive real axis.

We have considered the uncorrelated case thus far, but when X and Y are highly correlated, the difference of two RVs (i.e., X-Y) produces a distribution that is less skewed and has the properties of a normal distribution.

We use the following rules to determine the approximate shape of the resulting distribution:

1. If X has a larger variance than Y, then we expect X to dominate the variance of the distribution $X - Y$. The resulting distribution will have positive skewness.
   a. If $\sigma_X > \sigma_Y$, then $\kappa > 0$.
   b. Conversely, if $\sigma_X < \sigma_Y$, then $\kappa < 0$.
2. If the mean of X is larger than the mean of Y, the mean of $X - Y$ will be positive.
   a. If $\mu_X \geq \mu_Y$ and $\sigma_X > \sigma_Y$, then $X - Y$ will be approximately lognormally distributed.
   b. If $\mu_X \leq \mu_Y$ and $\sigma_X < \sigma_Y$, then $X - Y$ can be approximated by a negative-lognormal distribution.
3. If $\mu_X \leq \mu_Y$ and $\sigma_X > \sigma_Y$, then $X - Y$ can be approximated by a left-shifted lognormal distribution.
4. If $\rho_{X,Y}$ is large ($\rho_{X,Y} \sim 0.7$) or greater, then the distribution formed can be approximated by a normal distribution.

4.3.2 Covariance of Random Variables

When we are calculating the means and variances of CERs that rely on cost drivers that are random variables, we are interested in the functional transformation of the PDFs by the CER and the inclusion of the CER's error. To accurately calculate the moments of the CERs in the cost model, we must know how the CER and its error are correlated (or how they "covary") with each other in order to properly perform statistical summation.

Covariance is defined in Equation 4-16. Note that it is the expected value of the product of the differences of the random variables and their respective means. It is also defined in Equation 4-17 as the expected value of the product of the random variables minus the product of their means.

$$Cov(X,Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] \quad \textbf{4-16}$$

$$Cov(X,Y) = E[XY] - \mu_X \mu_Y, \text{ and} \quad \textbf{4-17}$$

$$Cov(X,X) = Var(X) = E[X^2] - E[X]^2$$

The correlation coefficient $\rho_{X,Y}$ in Equation 4-18 is the product-moment correlation coefficient, which relates $Cov(X,Y)$ to the product of the standard deviations of X and Y. This is the same Pearson product-moment correlation coefficient used in FRISK's statistical summation.

$$E[XY] = \rho_{X,Y} \sigma_X \sigma_Y + \mu_X \mu_Y \quad \textbf{4-18}$$

Two important theorems to remember are:

$$\text{If X,Y are independent, then } Cov(X,Y) = 0, \quad \textbf{4-19}$$

and the symmetry of covariance of Equation 4-20 requires us to only define the upper or lower off-diagonal elements of the correlation matrix (Figure 4-6), since $\rho_{i,j} = \rho_{j,i}$.

$$Cov(X, Y) = Cov(Y, X) \quad \textbf{4-20}$$

The bilinearity property of covariance means the following is true:

$$Cov(aX + b, cY + d) = acCov(X, Y) \quad \textbf{4-21}$$
$$Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y) \text{ and} \quad \textbf{4-22}$$
$$Cov(X, Y_1 + Y_2) = Cov(X, Y_1) + Cov(X, Y_2)$$

4.3.3 Transformation of Random Variables

When using linear CERs (and factors) such as $y = a + bX$, μ_X is shifted by the additive term (a) and scaled by the multiplicative term (b) (Equation 4-23), and the variance is scaled by the square of the multiplicative term (b) (Equation 4-24).

$$E(a + bX) = a + bE(X) = a + b\mu_X \quad \textbf{4-23}$$
$$Var(a + bX) = (b^2)Var(X) = b^2{\sigma_X}^2 \quad \textbf{4-24}$$

When linear transformations are applied to pairs of correlated random variables, the covariance is unaffected by the additive terms and is scaled by the multiplicative terms (Equation 4-25).

$$Cov(a + bX, c + dY) = (bd)Cov(X, Y) \quad \textbf{4-25}$$

We can calculate the correlation coefficient between two random variables, such as two CERs that share a common cost driver, using Equation 4-26.

$$Corr(X, Y) = \rho_{X,Y} = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}, \text{ and} \quad \textbf{4-26}$$
$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

To do this with a pair of CERs, we will need to determine the mean and sigma values for both CERs and the term $E[XY]$. The $E[XY]$ term is the expected value of the product of X and Y, which is why we call Pearson correlations “product-moment” correlations.

When nonlinear transformations are performed on random variables, as in the case where a CER, Y, is expressed as a function of a random variable, X:

$$Y = f(\mathrm{X}) = (a + b\mathrm{X}^c) \text{ ; where} \quad \textbf{4-27}$$
a, b, and c are coefficients of the CER with ($Var(\cdot) = 0$),

The terms $\mu_{Y,}\sigma_Y$ are computed as follows:

$$\mu_Y = \mu_{a+bX^c} = \mathrm{E}[a + bX^c] = a + bE[X^c] \quad \textbf{4-28}$$
$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{Var(a + bX^c)} \quad \textbf{4-29}$$

Since the variance of a constant is 0, $Var(a_{\mathrm{i}}) = 0$,

$$\sigma_Y = \sqrt{b^2 Var(X^c)} = b\sqrt{Var(X^c)}, \quad \textbf{4-30}$$

If $Z = x^c$ and $Var(\mathrm{Z}) = E[\mathrm{Z}^2] - E[Z]^2$ then

$$Var(X^c) = E[(X^c)^2] - (E[X^c])^2 = E[X^{2c}] - (E[X^c])^2 \quad \textbf{4-31}$$

The expectation $E[X^{\mathrm{k}}]$ is dependent on the shape of the probability distribution of X. In this case, if X is a triangular distribution, $X = T(L, M, H)$, then

$$E[X^{\mathrm{k}}] = \frac{2}{(H-L)(M-L)}\left\{\frac{M^{k+2}-L^{k+2}}{k+2} - L\frac{M^{k+1}-L^{k+1}}{k+1}\right\} + \frac{2}{(H-L)(H-M)}\left\{H\frac{H^{k+1}-M^{k+1}}{k+1} - \frac{H^{k+2}-M^{k+2}}{k+2}\right\}$$

Substituting k with c, we obtain:

$$E[X^c] = \frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{c+2}-L^{c+2}}{c+2} - L\frac{M^{c+1}-L^{c+1}}{c+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{c+1}-M^{c+1}}{c+1} - \frac{H^{c+2}-M^{c+2}}{c+2}\right\}\right]$$

and

$$\mu_Y = a + \frac{2b}{(H-L)(M-L)}\left\{\frac{M^{c+2}-L^{c+2}}{c+2} - L\frac{M^{c+1}-L^{c+1}}{c+1}\right\} + \frac{2b}{(H-L)(H-M)}\left\{H\frac{H^{c+1}-M^{c+1}}{c+1} - \frac{H^{c+2}-M^{c+2}}{c+2}\right\}$$

So $\frac{\mu_Y - a}{b} = E[X^c]$ and $Var(X^c)$ can be rewritten as:

$$Var(X^c) = E[X^{2c}] - \left(\frac{\mu_f - a}{b}\right)^2$$

$$Var(X^{2c}) = \frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{2c+2}-L^{2c+2}}{2c+2} - L\frac{M^{2c+1}-L^{2c+1}}{2c+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{2c+1}-M^{2c+1}}{2c+1} - \frac{H^{2c+2}-M^{2c+2}}{2c+2}\right\}\right] - \left(\frac{\mu_f - a}{b}\right)^2$$

Using Equation 4-30,

$$\sigma_Y = b\sqrt{\frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{2c+2}-L^{2c+2}}{2c+2} - L\frac{M^{2c+1}-L^{2c+1}}{2c+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{2c+1}-M^{2c+1}}{2c+1} - \frac{H^{2c+2}-M^{2c+2}}{2c+2}\right\}\right] - \left(\frac{\mu_f - a}{b}\right)^2}$$

This is a rather lengthy equation, so VBA expressions are provided in Appendix D.

From this point forward, where a VBA function exists, such as for $E[X^{\mathrm{k}}]$, we will leave any expansions of equations in terms of $E[X^{\mathrm{k}}]$.

We often rely on the calculation of the higher-order moments to determine probability distributions used in estimating relationships. The k^{th} moment of the RV X is

$$E[X^k] = \begin{cases} \sum_X x^k P_X(x) & ; if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^k f_X(x) dx & ; if\ X\ is\ continuous \end{cases} \quad \textbf{4-32}$$

In summary, we can use the equations for expected value, variance, and covariance to find the moments of a distribution and the covariance (and correlation between random variables). Another simpler way of dealing with complex transformations of independent random variables is through the use of Mellin transforms (Section 6).

4.3.4 Multiplication and Division of Random Variables

Often, we are interested in the moments of the PDF of the product or transformation of multiple random variables in an equation such as a CER. Three methods of finding the moments in this situation are the use of: 1) expectation operations, 2) Mellin transforms and 3) propagation of errors. The first method is an extension of the expectation operations shown in Section 4.3.2, and the last two methods are discussed in greater detail in Sections 6 and 7. Section 5 provides a general formula for the variance of the product of two or more random variables.

5 Product of Dependent Random Variables

The moments of the PDF formed by the product of two dependent random variables are used frequently in probabilistic cost analysis. Products of random variables are found in probabilistic cost estimates using CERs that have correlated error terms, or when using cost-dependent CERs. Products of multiple random variables occur when calculating the correlation coefficient between different WBS elements. We first provide equations for the moments of the product of two jointly normal random variables, then follow with the case in which we have two jointly lognormal random variables. Using the methods used to derive these equations, we provide equations for the moments of the product of multiple random variables.

5.1 Product of Two Normal Random Variables

In the first case, we derive the moments of the product of two random variables that are defined using normal PDFs. If X and Y are jointly dependent random variables defined by:

$X = \mu_X + r\sigma_X Z + \sqrt{1-r^2}\sigma_X E_1$, and $Y = \mu_Y + r\sigma_Y Z + \sqrt{1-r^2}\sigma_Y E_2$

where Z, E_1, and E_2 are independent, standard normal PDFs (i.e., $N(0,1)$), then their covariances are zero. This means $Cov(Z, E_1) = 0$, $Cov(Z, E_2) = 0$, and $Cov(E_1, E_2) = 0$. We can further state the means of X and Y are $E[X] = \mu_X$, $E[Y] = \mu_Y$. The variances of X and Y are $Var(X) = \sigma_X{}^2, Var(Y) = \sigma_Y{}^2$. Finally, we define $c = Cov(X,Y) = r^2\sigma_X\sigma_Y$. $r^2 = \rho_{X,Y}$ by definition.

The expected value of the product XY is:

$E[XY] = Cov(X,Y) + E[X]E[Y] = \rho_{X,Y}\sigma_X\sigma_Y + \mu_X\mu_Y$ using Equation 4-18.

The variance of the product is found through some manipulation:

$$Var[XY] = E[(XY)^2] - E^2[XY]$$

$$E[(XY)^2] = Cov(X^2, Y^2) + E[X^2]E[Y^2]$$

$$Var[XY] = Cov(X^2, Y^2) + E[X^2]E[Y^2] - (Cov(X,Y) + E[X]E[Y])^2$$

$$Var[XY] = Cov(X^2, Y^2) + E[X^2]E[Y^2] - (Cov^2(X,Y) + 2E[X]E[Y]Cov(X,Y) + E^2[X]E^2[Y])$$

$$Var[XY] = Cov(X^2, Y^2) + E[X^2]E[Y^2] - Cov^2(X,Y) - 2E[X]E[Y]Cov(X,Y) - E^2[X]E^2[Y]$$

$E[X^2] = \mu_X{}^2 + \sigma_X{}^2$ and $E[Y^2] = \mu_Y{}^2 + \sigma_Y{}^2$

$Var[XY] = Cov(X^2, Y^2) + ({\mu_X}^2 + {\sigma_X}^2)({\mu_Y}^2 + {\sigma_Y}^2) - Cov^2(X,Y) - 2\mu_X\mu_Y r^2 \sigma_X \sigma_Y - {\mu_X}^2{\mu_Y}^2$

$Var[XY] = Cov(X^2, Y^2) + {\mu_X}^2{\mu_Y}^2 + {\mu_X}^2{\sigma_Y}^2 + {\sigma_X}^2{\mu_Y}^2 + {\sigma_X}^2{\sigma_Y}^2 - c^2 - 2\mu_X\mu_Y c - {\mu_X}^2{\mu_Y}^2$

$$Var[XY] = Cov(X^2, Y^2) + {\mu_X}^2{\sigma_Y}^2 + {\sigma_X}^2{\mu_Y}^2 + {\sigma_X}^2{\sigma_Y}^2 - c^2 - 2\mu_X\mu_Y c \quad \textbf{5-1}$$

This is the same result obtained by (Goodman, L. A., 1960) and (Bohrnstedt & Goldberger, 1969).[40, 41]

To solve the $Cov(X^2, Y^2)$ term, we must expand the squares of X and Y, use the definition of covariance provided in Equation 4-17, and insert that result into Equation 5-1. This derivation is provided in Appendix C – Derivations, Section 16.3.7. The resulting covariance term is

$Cov(X^2, Y^2) = 4\mu_X\mu_Y c + 2c^2$

This allows us to express the variance of the product of two normally distributed PDFs as:

$Var[XY] = 4\mu_X\mu_Y c + 2c^2 + {\mu_X}^2{\sigma_Y}^2 + {\sigma_X}^2{\mu_Y}^2 + {\sigma_X}^2{\sigma_Y}^2 - (c)^2 - 2\mu_X\mu_Y c$

This simplifies to Equation 5-2.

$$Var[XY] = 2\mu_X\mu_Y c + c^2 + {\mu_X}^2{\sigma_Y}^2 + {\sigma_X}^2{\mu_Y}^2 + {\sigma_X}^2{\sigma_Y}^2 \quad \textbf{5-2}$$

When X and Y are independent, $c = 0$, Equation 5-2 reduces to Equation 5-3.

$$Var[XY] = {\mu_X}^2{\sigma_Y}^2 + {\sigma_X}^2{\mu_Y}^2 + {\sigma_X}^2{\sigma_Y}^2 \quad \textbf{5-3}$$

When $Y = X$, $c = {\sigma_X}^2$, Equation 5-2 becomes Equation 5-4.

$$Var[X^2] = 2{\sigma_X}^2(2{\mu_X}^2 + {\sigma_X}^2) \quad \textbf{5-4}$$

5.2 Product of Two Lognormal PDFs

In the case where we are interested in the product of two lognormal PDFs, we cannot rely on the symmetric properties of the normal distribution to cancel terms and also cannot rely

[40] Goodman, L. A. (1960, Dec.). On the Exact Variance of Products. Journal of the American Statistical Association, 55(292), 708-713.

[41] Bohrnstedt, G. W., & Goldberger, A. S. (1969, Dec.). On the Exact Covariance of Products of Random Variables. Journal of the American Statistical Association, 64(328), 1439-1442.

on the standard normal distributions zero-mean properties to manipulate the equations. We must rely on the fact that the lognormal distribution is related to the exponent of an underlying normal distribution.

If X_1 and X_2 are jointly distributed normal random variables with ρ_{X_1,X_2}, then Y_1 and Y_2 are jointly distributed lognormal random variables with ρ_{Y_1,Y_2}, and $Y_1 = e^{X_1}$, and $Y_2 = e^{X_2}$. If X_1 and X_2 are defined by $N(P_1, Q_1)$, and $N(P_2, Q_2)$, then Y_1 and Y_2 are defined by $L(\mu_{Y_1}, \sigma_{Y_1})$, and $L(\mu_{Y_2}, \sigma_{Y_2})$, respectively.[42] The mean and variance of Y_1 and Y_2 are:

$\mu_{Y_i} = e^{\left(P_i + \frac{1}{2}Q_i^2\right)}$ and $\sigma_{Y_i}^2 = e^{(2P_i + Q_i^2)}\left(e^{Q_i^2} - 1\right)$ and

$$\rho_{X_1,X_2} = \frac{1}{Q_1 Q_2}\ln\left[1 + \rho_{Y_1,Y_2}\left(\sqrt{e^{Q_1^2} - 1}\sqrt{e^{Q_2^2} - 1}\right)\right].$$

The product $Z = Y_1 Y_2 = e^{X_1} e^{X_2} = e^{X_1 + X_2}$, so the distribution of $\ln(Z)$ has mean:

$E[\ln(Z)] = P_1 + P_2$, and variance, $[\ln(Z)] = Q_Z^2 = Q_1^2 + 2\rho_{X_1,X_2} Q_1 Q_2 + Q_2^2$.

Therefore, the mean and variance of $Z = Y_1 Y_2$ is:

$$\mu_Z = e^{\left([P_1+P_2] + \frac{1}{2}[Q_1^2 + 2\rho_{X_1,X_2} Q_1 Q_2 + Q_2^2]\right)}, \text{ and} \quad \textbf{5-5}$$

$$\sigma_Z^2 = e^{(2[P_1+P_2] + [Q_1^2 + 2\rho_{X_1,X_2} Q_1 Q_2 + Q_2^2])}\left(e^{[Q_1^2 + 2\rho_{X_1,X_2} Q_1 Q_2 + Q_2^2]} - 1\right) \quad \textbf{5-6}$$

Equation 5-5 is an exact solution of the variance of the product of two lognormal distributions. Results of the exact standard deviation using the square-root of the variance calculation using Equation 5-5 are compared to a 100,000-trial statistical simulation in Table 5-1. The simulated mean of the product is low compared to the exact result due to the inability to correlate the two RVs to exactly $\rho = 0.5$. The simulated standard deviation is slightly lower than the exact result due to uneven sampling of the lognormal PDFs.

Table 5-1 Analytic and Simulated Results of the Product of Two Lognormal PDFs

	Analytic			**Simulated**		
	μ	σ	ρ_{Y_1,Y_2}	μ	σ	ρ_{Y_1,Y_2}
Y_1	1.000	1.000	0.500	0.999	0.999	0.432
Y_2	1.000	1.000		0.999	0.999	
$Y_1 Y_2$	1.500	4.243		1.430	3.749	

[42] Lognormal Distributions: Theory and Applications
Edwin L. Crow, Kunio Shimizu, 1988. Marcel Dekker, NY, Statistics, textbooks and monographs Series, vol. 88, p14-17.

When Y_1 and Y_2 are independent, $\rho_{Y_1,Y_2} = 0$, so the mean and variance of Z are:

$$\mu_Z = e^{\left([P_1+P_2]+\frac{1}{2}[Q_1^2+2Q_1Q_2+Q_2^2]\right)}, \text{ and} \quad \textbf{5-7}$$
$$\sigma_Z^2 = e^{(2[P_1+P_2]+[Q_1^2+Q_2^2])}\left(e^{[Q_1^2+Q_2^2]} - 1\right) \quad \textbf{5-8}$$

To calculate the moments of the square of Y_1, we can set $Y_1 = Y_2$, so $\mu_{Y_1} = \mu_{Y_2}$, $\rho_{Y_1,Y_2} = 1$. The resulting mean and variance of Z are:

$$\mu_Z = e^{2(P_1+2Q_1^2)}, \text{ and} \quad \textbf{5-9}$$
$$\sigma_Z^2 = e^{(2P_1+4Q_1^2)}\left(e^{[4Q_1^2]} - 1\right) \quad \textbf{5-10}$$

Additionally, when $\mu_{Y_1} = 1$, and $\sigma_{Y_1} = 1$ (i.e., Y_1 is a unit lognormal distribution, $Y_1 = L(1,1)$), then $Var\left[{Y_1}^2\right] = 60$.

Since $\sigma_{Y_1} = \sqrt{Var\left[{Y_1}^2\right]}$, $\sigma_{Y_1} = \sqrt{60}$, or 7.7459667.

Comparing these results to a statistical simulation, we get similar means but different standard deviations as shown in Table 5-2.

Table 5-2 Analytic and Simulated Results of the Square of Two Lognormal PDFs

	Analytic		Simulated	
	μ	σ	μ	σ
Y_1	1.000	1.000	1.000	1.005
${Y_1}^2$	2.000	7.746	2.010	8.900

The difference between the sigma values from the analytic (exact) answer and the simulated (approximate) answer is due to the simulation's sampling of the lognormal PDF. Since none of the error can be attributed to the correlation between random variables (i.e., it is a square of a single RV), it must be due to the ability of the simulation to sample the large tails of the lognormal PDFs. Looking at the results of the variance from 10 simulation runs of 100,000 trials each shows the simulated variance is biased low and there is a large standard deviation of results of the variance of ${Y_1}^2$. This is due to the fact that sampling highly skewed distributions will always be difficult for simulations, so simulations cannot always be trusted in these situations. It is best to check your simulation's results to see that the simulation has reproduced the correct Pearson correlation coefficient and that the means and standard deviations of the inputs and product are correctly computed.

Table 5-3 Ten Simulated Sample Runs of Variance of LN PDF Squared

Simulation Run	$Var({Y_1}^2)$	Simulation Run	$Var({Y_1}^2)$
1	49.894	6	63.005
2	54.359	7	58.854
3	47.536	8	57.698
4	51.769	9	57.165
5	87.246	10	49.030
$\mu_{Var({Y_1}^2)}$	57.656		
$\sigma_{Var({Y_1}^2)}$	11.491		

5.3 Product of Exponentiated Lognormal PDFs

In some cases, it may become necessary to calculate the product of two lognormal PDFs that are exponentiated. Exponentiation of a lognormal PDF Y_1 by some constant exponent, c, (i.e., Y_1^c) is equivalent to multiplying its underlying normal distribution by c.

$$Y_1^c = e^{cX_1}$$

If the distribution X_1 has mean P_1 and standard deviation Q_1, then the distribution cX_1 will have mean cP_1 and standard deviation cQ_1. If we multiply two exponentiated lognormal PDFs Y_1 and Y_2 by exponents c and d, we can compute the mean and variance of the resulting distribution, $Z = Y_1^c Y_2^d$, using the exponents of the underlying normal distributions of Y_1 and Y_2, which are X_1 and X_2.

$$Z = Y_1^c Y_2^d = e^{cX_1} e^{dX_2} = e^{(cX_1 + dX_2)}$$

With the mean and variance of the underlying normal distribution,

$P_Z = cP_{X_1} + dP_{X_2}$ and $Q_Z^2 = c^2 Q_{X_1}^2 + 2\rho_{X_1,X_2} cd Q_{X_1} Q_{X_2} + d^2 Q_{X_2}^2$

the correlation between the underlying normal PDFs, ρ_{X_1,X_2}, will be unaffected by the affine transformation[43] of the underlying normal distribution. The correlation between the lognormal PDFs, ρ_{Y_1,Y_2}, will also remain unchanged. The correlation between the variables U and V ($\rho_{U,V}$), where $U = Y_1^c$ and $V = Y_2^d$, will be different from that of ρ_{Y_1,Y_2}, however.

[43] An affine transformation does not change the properties of the variable(s) undergoing the transformation. For example, the correlation between two RVs is unchanged when either (or both) undergo a linear transformation. That linear transformation is considered an affine transformation.

5.3.1 Correlation Between Exponentiated Lognormal PDFs

Using the derivation above, the exponentiated lognormal RVs undergo a non-affine transformation, meaning their relationship to each other changes. In the case of the product of lognormal RVs, $Z = UV = Y_1^c Y_2^d$, the correlation $\rho_{U,V}$ is calculated using:

$$\rho_{U,V} = \frac{e^{\left(cd\rho_{X_1,X_2} Q_{X_1} Q_{X_2}\right)} - 1}{\sqrt{e^{\left(cQ_{X_1}\right)^2} - 1}\sqrt{e^{\left(dQ_{X_2}\right)^2} - 1}}$$

As an example, We will exponentiate two lognormal PDFs (Y_1 and Y_2) defined by $L(1{,}0.5)$ with correlation $\rho_{Y_1,Y_2} = 0.5$. We wish to find the correlation, $\rho_{U,V}$, where $U = Y_1^c$, $V = Y_2^d$, $c = 0.9$, and $d = 1.2$. First we must find Q_1 and Q_2 where:

$Q_i = \sqrt{\ln\left[\frac{\mu_{Y_i}{}^2 + \sigma_{Y_i}{}^2}{\mu_{Y_i}{}^2}\right]}$, which results in $Q_1 = 0.4724$ and $Q_2 = 0.4724$.

Next we calculate ρ_{X_1,X_2} using $\rho_{X_1,X_2} = \frac{1}{Q_1 Q_2}\ln\left[1 + \rho_{Y_1,Y_2}\left(\sqrt{e^{Q_1^2} - 1}\sqrt{e^{Q_2^2} - 1}\right)\right]$.

$$\rho_{X_1,X_2} = \frac{1}{(0.4724)(0.4724)}\ln\left[1 + (0.5)\left(\sqrt{1.25 - 1}\sqrt{1.25 - 1}\right)\right] = 0.5278$$

Last, we have the correlation between U and V:

$$\rho_{U,V} = \frac{e^{([0.9][1.2][0.5278][0.4724][0.4724])} - 1}{\sqrt{e^{([0.9][0.4724])^2} - 1}\sqrt{e^{([0.4724][0.4724])^2} - 1}} = 0.4951.$$

5.4 Product of Multiple Lognormal PDFs

In the case where cost-on-cost factors are used in a probabilistic cost estimate, the correlation between a WBS element that is estimated using a cost-on-cost factor and its base is governed by the expected value of the product of multiple random variables.

We use the case where we have three random variables representing the multiplicative uncertainties of three CERs, ε_1, ε_2 and ε_3. The products used in the correlation matrix may include the following terms: $\varepsilon_1\varepsilon_2\varepsilon_3$, $\varepsilon_1{}^2\varepsilon_2$, $\varepsilon_1{}^2\varepsilon_2\varepsilon_3$, among others.

The expectation of any combination or exponentiation of products of $\varepsilon_1, \varepsilon_2,$ or ε_3 is derived using a set of jointly dependent lognormally distributed PDFs defined by their respective means and variances. In the case of the triple product $Z = \varepsilon_1\varepsilon_2\varepsilon_3$, the mean of the underlying normal distribution formed by the triple product is:

$E[ln(Z)] = \sum P_i$

and the variance is

$Var[ln(Z)] = \sum Q_j^{\ 2} + \sum_{i \neq k} \sum_k \rho_{X_i,X_k} Q_i Q_k$, where

$\rho_{X_i,X_k} = \frac{1}{Q_i Q_k} \ln\left[1 + \rho_{\varepsilon_i,\varepsilon_k}\left(\sqrt{e^{Q_i^2} - 1}\sqrt{e^{Q_k^2} - 1}\right)\right]$

$$\mu_Z = e^{\left(\sum P_i + \frac{1}{2}\sum Q_j^{\ 2} + \sum_{i \neq k}\sum_k \rho_{X_i,X_k} Q_i Q_k\right)}, \text{ and} \qquad \textbf{5-11}$$

$$\sigma_Z^2 = e^{\left(2\sum P_i + \sum Q_j^{\ 2} + \sum_{i \neq k}\sum_k \rho_{X_i,X_k} Q_i Q_k\right)}\left(e^{\sum Q_j^{\ 2} + \sum_{i \neq k}\sum_k \rho_{X_i,X_k} Q_i Q_k} - 1\right) \qquad \textbf{5-12}$$

5.5 Limitations of Statistical Simulations

Statistical simulations, due to their inability to perfectly sample correlated random variables will produce some error, of course. To test these errors, we defined three lognormally distributed random variables $\varepsilon_1, \varepsilon_2,$ and $\varepsilon_3$with a lognormal PDF, $L(1{,}0.5)$, and defined their inter-element correlation, $\rho_{\varepsilon_i,\varepsilon_k} = 0.5$. We then calculated the expectations of the products discussed above using the analytic method and with a 100,000-trial statistical simulation. The results are shown in Table 5-4. Over the 10 different simulation runs, the average of the means (1.414) was less than that of the analytic (true) result (1.424). Also, the average of the variances from the 10 runs (5.776) was less than that of the analytic (true) result (6.000). The simulations produced a wide range of variances represented by the standard deviation of the simulated variance results (0.229).

Table 5-4 Ten Simulated Sample Runs of Variance of Triple Product of LN PDF

Simulation Run	$E(Z)$	$Var(Z)$	Simulation Run	$E(Z)$	$Var(Z)$
1	1.409	5.435	6	1.416	6.100
2	1.412	5.534	7	1.410	5.593
3	1.418	5.923	8	1.411	5.573
4	1.417	6.053	9	1.417	5.818
5	1.415	5.880	10	1.413	5.853
Average	1.414	5.776	**Analytic**	1.424	6.000
Std. Dev.	0.003	0.229			

6 Mellin Transforms

A Mellin transform is a type of integral transform that allows us to find the moments of user-specified random variables or functions of random variables, such as CERs. This is particularly useful in uncertainty analysis because we often need to find the moments of 1) the product of two or more independent random variables, and 2) transformations of random variables (e.g., exponentiation).

As with anything that looks "too good to be true", there are restrictions on its use. We will first define Mellin transforms, show how to use them and provide an example. The Mellin Transform[44, 45] of a function *f(X)*, where *X* is a positive random variable, is defined as:

$$\mathcal{M}_X(s) = \mathcal{M}[f(X); s] = \int_0^\infty x^{s-1} f(x)dx, x > 0, \text{ where} \quad \textbf{6-1}$$

$\mathcal{M}_X(s)$ is the Mellin transform of $f(X)$, and
s is the order of the transform

As with the Fourier and Laplace transforms, there is a one-to-one correspondence between $M_X(s)$ and $f(X)$. When $f(X)$ is a PDF, we can see the relationship between the Mellin transform of a PDF and the moments about the origin μ' as:

$$\mu_{s-1}{}' = E[X^{s-1}] = \mathcal{M}_X(s) \quad \textbf{6-2}$$

6.1 Mellin Transform Properties

Mellin transforms allow us to calculate moments of results of operations on independent random variables. Table 6-1 shows the Mellin transforms of simple operations on single independent random variables.

Table 6-1 Operation Properties of Mellin Transform on a PDF

	Property	PDF	RV	Mellin Transform
a.	Standard	$f(x)$	X	$\mathcal{M}_X(s)$
b.	Scaling	$f(ax)$	X	$a^{(-s)}\mathcal{M}_X(s)$
b.	Linear	$af(x)$	X	$a\mathcal{M}_X(s)$
d.	Translation	$x^a f(x)$	X	$\mathcal{M}_X(a+s)$
e.	Exponentiation	$f(x^a)$	X	$a^{(-1)}\mathcal{M}_X(s/a)$

Table 6-2 shows the Mellin transforms of more complex operations on single and multiple independent random variables.

[44] Giffin, W.C., *Transform Techniques for Probability Modeling*, Academic Press, 1975.
[45] Springer, M.D., *The Algebra of Random Variables*, John Wiley and Sons, 1979.

Table 6-2 Mellin Transform of Products and Quotients of Random Variables

	Random Variable	PDF Given	$\mathcal{M}_Z(s) =$
a.	$Z=X$	$f(x)$	$\mathcal{M}_X(s)$
b.	$Z=X^b$	$f(x)$	$\mathcal{M}_X(bs-b+1)$
c.	$Z=1/X$	$f(x)$	$\mathcal{M}_X(2-s)$
d.	$Z=XY$	$f(x), g(y)$	$\mathcal{M}_X(s)\mathcal{M}_Y(s)$
e.	$Z=X/Y$	$f(x), g(y)$	$\mathcal{M}_X(s)\mathcal{M}_Y(2-s)$
f.	$Z=aX^bY^c$	$f(x), g(y)$	$a^{(s-1)}\mathcal{M}_X(bs-b+1)\mathcal{M}_Y(cs-c+1)$

6.2 Mellin Transform of the Uniform Distribution

The uniform distribution, $U(L,H)$, has a PDF defined by:

$$f(x) = 1/(H-L);\ L \le x \le H, \qquad \textbf{6-3}$$

and a Mellin transform defined by

$$\mathcal{M}[f(x);s] = \frac{(H^s - L^s)}{s(H-L)} \qquad \textbf{6-4}$$

6.3 Mellin Transform of the Triangular Distribution

The triangular distribution, $T(L,M,H)$, has a PDF defined by:

$$f(x) = \begin{cases} \frac{2(x-L)}{(H-L)(M-L)} & ;0 < L < x \le M \\ \frac{2(H-x)}{(H-L)(H-M)} & ;M \le x \le H \end{cases} \qquad \textbf{6-5}$$

and a Mellin transform defined by

$$\mathcal{M}[f(x);s] = \frac{2}{[(H-L)s(s+1)]}\left\{\frac{H(H^s-M^s)}{(H-M)} - \frac{L(M^s-L^s)}{(M-L)}\right\} \qquad \textbf{6-6}$$

6.4 Mellin Transform Example

In this example, we will apply Mellin transforms to a multivariate CER[46] with error:

$Y = aX_1{}^bX_2{}^c\ \varepsilon$, where **6-7**
Y is cost, a random variable (RV)
$a, b,$ and c are constants, $a = 0.1, b = 0.95,$ and $c = 0.60$
X_1 is a cost driver that is a RV, $X_1 = \mathrm{T}(9,10,15)$
X_2 is a cost driver that is a RV, $X_2 = \mathrm{T}(30,40,60)$

[46] The CER's cost drivers and inputs are uncorrelated (all $\rho_{i,j} = 0$).

ε is the percent standard error of the CER, a RV, $\varepsilon = \mathrm{N}(1{,}0.3)$

This CER has two cost drivers that are random variables (X_1 and X_2) and a CER standard percent error, ε. We will split the problem into pieces; one piece will be the term $f(x) = a{X_1}^b{X_2}^c$, and the other will be the error term, ε.

Remember, when $s = 2$, we are calculating the first moment (mean) and when $s = 3$ we are calculating the second raw moment (i.e., about the origin) and have to correct for the mean to get the second moment about the mean.

To solve this problem, we will follow these steps:

1. Find the appropriate Mellin transforms of a PDF (Equation 6-6)
2. Calculate the Mellin transforms for each operation as shown in Table 6-1 and Table 6-2.
3. Determine the mean and sigma values from the Mellin transform

In the first step, we need to find the Mellin transform of $f(x)$ and ε for orders $s = 2$ and $s = 3$, then apply the rule from multiplying RVs $f(x)$ and ε.

Let us begin with defining $M[f(x); s]$ for X_1, which is a triangular distribution, so:

$$\mathcal{M}[X_1\,; s] = \mathcal{M}[\mathrm{T(L,M,H)}; s] = \frac{2}{[(H-L)s(s+1)]}\left\{\frac{H(H^s-M^s)}{(H-M)} - \frac{L(M^s-L^s)}{(M-L)}\right\}$$

We must now find $\mathcal{M}[f(x); 2]$and $\mathcal{M}[f(x); 3]$, where $f(x) = \; a{X_1}^b{X_2}^c$. From Table 6-2,

$\mathcal{M}[f(x); s] = a^{(s-1)}\mathcal{M}_{X_1}\,(bs - b + 1)\mathcal{M}_{X_2}(cs - c + 1)$, where b=0.95 and c=0.6
$\mathcal{M}[f(x); s] = a^{(s-1)}\mathcal{M}_{X_1}\,(0.95s - 0.95 + 1)\mathcal{M}_{X_2}(0.6s - 0.6 + 1)$
$\mathcal{M}[f(x); s]$= $a^{(s-1)}\mathcal{M}_{X_1}\,(0.95s + 0.05)\mathcal{M}_{X_2}(0.6s + 0.4)$

For $s = 2$, $\mathcal{M}_{X_1}\,(0.95s + 0.05) = \mathcal{M}_{X_1}\,(1.95) = \frac{2}{[(H-L)(1.95)(2.95)]}\left\{\begin{matrix}\frac{H\left(H^{(1.95)}-M^{(1.95)}\right)}{(H-M)} \\ -\frac{L\left(M^{(1.95)}-L^{(1.95)}\right)}{(M-L)}\end{matrix}\right\}$

$$\mathcal{M}[X_1\,; 1.95] = \frac{2}{[(15-9)(1.95)(2.95)]}\left\{\frac{15\left(15^{1.95}-10^{1.95}\right)}{(15-10)} - \frac{9\left(10^{1.95}-9^{1.95}\right)}{(10-9)}\right\} = 10.035$$

Using the same formula, for order $s = 2.95$,

$\mathcal{M}[X_1\,; 2.95] = \mathcal{M}[\mathrm{T}(9{,}10{,}15); 2.95] = 101.911$.

Since $\mathcal{M}_{X_2}\,(0.6s + 0.4)$, we have to find $\mathcal{M}[X_2; 1.6]$, and $\mathcal{M}[X_2; 2.2]$. X_2 is a PDF defined by a triangular distribution, $T(30{,}40{,}60)$, so

$\mathcal{M}[X_2; 1.6] = \mathcal{M}[T(30{,}40{,}60); 1.6] = 9.572$, and $\mathcal{M}[X_2; 2.2] = \mathcal{M}[T(30{,}40{,}60); 2.2] = 92.312$.

Now we can multiply the terms to find $\mathcal{M}[f(x);2]$and $\mathcal{M}[f(x);3]$

$\mathcal{M}[f(x);2] = a^{(1)}\mathcal{M}_{X_1}(1.05)\mathcal{M}_{X_2}(1.6) = (0.1)(10.035\,)(101.911) = 9.606$, and
$\mathcal{M}[f(x);3] = a^{(2)}\mathcal{M}_{X_1}(3)\mathcal{M}_{X_2}(3) = (0.01)(9.572)(92.312) = 94.076.$

The mean and sigma of $f(x)$ are:

$\mu_{f(x)} = \mathcal{M}[f(x);2] = 9.606,$
$Var\big(f(x)\big) = \mathcal{M}[f(x);3] – (\mathcal{M}[f(x);2])^2 = 94.076 - (9.606)^2 = 1.8089$, and
$\sigma_{f(x)} = \sqrt{Var\big(f(x)\big)} = \sqrt{1.8089} = 1.345.$

Finally, we have to calculate the Mellin transformation of ε to complete our example problem. Unfortunately, the Mellin transform for a normal distribution is *not* defined over the entire range, only from 0 to +∞ (i.e., non-negative values), so we must find a way to overcome this limitation. But fortunately, we already know the mean and sigma of ε and can “back out” $\mathcal{M}(\varepsilon;2)$ and $\mathcal{M}(\varepsilon;3)$.

We already know the mean and sigma of ε by its definition as the multiplicative standard error, $N(1,0.3)$. Given this information,

$\mathcal{M}(\varepsilon;2) = \mu_\varepsilon = 1.0$, and
$\mathcal{M}[\varepsilon;3] = Var(\varepsilon) + (\mathcal{M}[\varepsilon,2])^2 = \sigma_\varepsilon{}^2 + \mu_\varepsilon{}^2 = (1^2) + (0.3^2) = 1.09.$

From Table 6-2, $\mathcal{M}[Y\varepsilon;s] = \mathcal{M}[Y;s]\mathcal{M}[\varepsilon;s]$, so:

$\mathcal{M}[Y;2] = \mathcal{M}[f(x);2]\mathcal{M}[\varepsilon;2] = (9.606)(1) = 9.606$, and
$\mathcal{M}[Y;3] = \mathcal{M}[f(x);3]\mathcal{M}[\varepsilon;3] = (94.076)(1.09) = 102.543.$

The exact mean and sigma values are:

$\mu_{(Y\varepsilon)} = \mathcal{M}[Y;2] = 9.606,$
$\sigma_{(Y\varepsilon)} = \sqrt{\mathcal{M}[Y;3] – (\mathcal{M}[Y,2])^2} = \sqrt{102.543 - (9.606)^2} = \sqrt{10.276} = 3.206.$

The mean and standard deviation from a 100,000-trial statistical simulation using the parameters specified in Equation 6-7 result in:

$\hat{\mu}_{(Y\varepsilon)} = 9.60$, and $\hat{\sigma}_{(Y\varepsilon)} = 3.19$

Since the Mellin transform method provides the exact value, the differences are due to simulation errors. Indeed, a dump of the trial values for $\boldsymbol{X_1}, \boldsymbol{X_2}$, and $\boldsymbol{\varepsilon}$ followed by a calculation of their inter-element correlations reveals that $\boldsymbol{\rho} \neq \boldsymbol{I}$ (i.e., the correlation matrix does not equal the identity matrix) as shown in Table 6-3. This means some of the error in the simulation is due to its inability to sample (un)correlated random variables.

Table 6-3 Correlation Coefficients from 100,000-Trial Statistical Simulation

	ε	X_1	X_2
ε	1.0000	-0.0031	-0.0111
X_1	-0.0031	1.0000	-0.0038
X_2	-0.0111	-0.0038	1.0000

7 Propagation of Errors

Cost analysts often need to find the moments of the product of two uncorrelated random independent variables such as a CER and its percent error.[47] For example,

$Y = f(x)\varepsilon$; where

x is a random variable describing the input (e.g., weight)
$f(x)$ is an estimating relationship with x as an independent variable
ε is a random variable describing the estimating error

The "Propagation of Errors" method allows us to calculate the mean and sigma values of the product of two uncorrelated random variables A and B.[48] Proof of this is provided in Appendix C – Derivations.

$$\mu_{AB} = \mu_A \mu_B \qquad \textbf{7-1}$$
$$\sigma_{AB} = \sqrt{(\mu_A \sigma_B)^2 + (\sigma_A \mu_B)^2 + (\sigma_A \sigma_B)^2} \qquad \textbf{7-2}$$

For our example problem, we will break the CER and its error into two parts, A and B, where $A = f(x)$ and $B = \varepsilon$. In this case,

$$\mu_{AB} = \mu_{f(x)} \mu_\varepsilon \qquad \textbf{7-3}$$
$$\sigma_{AB} = \sqrt{\left(\mu_{f(x)} \sigma_\varepsilon\right)^2 + \left(\sigma_{f(x)} \mu_\varepsilon\right)^2 + \left(\sigma_{f(x)} \sigma_\varepsilon\right)^2} \qquad \textbf{7-4}$$

Since the multiplicative error has a mean, $\mu_\varepsilon = 1$, and the standard deviation of the error is predefined, the equation reduces to

$$\mu_{AB} = \mu_{f(x)} \qquad \textbf{7-5}$$
$$\sigma_{AB} = \sqrt{\left(\mu_{f(x)} \sigma_\varepsilon\right)^2 + \left(\sigma_{f(x)}\right)^2 + \left(\sigma_{f(x)} \sigma_\varepsilon\right)^2} \qquad \textbf{7-6}$$

Previously, we showed how to statistically sum random variables using FRISK. Now we will show how to perform other operations such as multiplying random variables. This type of operation is particularly necessary when we need to calculate the uncertainty of CERs that have multiplicative standard errors. The propagation of errors allows us to do this in a clean, straightforward manner.

[47] The random variables representing a CER and its multiplicative error should be uncorrelated.
[48] *Engineering Statistics Handbook*, National Institute of Standards, Section 2.5.5

7.1 Propagation of Errors Example

For our example, we will estimate the μ, σ and 70th percentile of total cost using the three point estimates (originally from the FRISK example from Book (1994) in Table 4-5) and estimating errors in Table 7-1. In this example, estimates are random variables defined by triangular distributions, and CER errors are either normal or lognormal random variables with $\mu_{\varepsilon_i} = 1$.

Table 7-1 Propagation of Errors Example

WBS Element, i	Estimate, $f(x)_i$	CER Error, ε_i
Antenna	T(191,380,1151)	N(1,0.20)
Electronics	T(96,192,582)	L(1,0.31)
Platform	T(33,76,143)	L(1,0.40)
Facilities	T(9,18,27)	N(1,0.20)
Power Distribution	T(77,154,465)	N(1,0.35)
Computers	T(30,58,86)	N(1,0.30)
Environmental Control	T(11,22,66)	L(1,0.30)
Communications	T(58,120,182)	N(1,0.30)
Software	T(120,230,691)	L(1,0.30)

To demonstrate this method, we will perform an example calculation using the first WBS element. The Antenna WBS element CER is defined by a triangular distribution, $T(191{,}380{,}1151)$. Using the calculations from our FRISK example in Table 4-5, $\mu_{f(x)_1} = 574$, and $\sigma_{f(x)_1} = 207.62$. The Antenna CER has a standard error, ε_1 defined by a normal distribution, $N(1{,}0.20)$, so $\mu_{\varepsilon_1} = 1$, and $\sigma_{\varepsilon_1} = 0.2$. Using the propagation of errors equations (7-5 and 7-6),

$$\mu_{AB} = \mu_{f(x)_1}\mu_{\varepsilon_1} = (574)(1) = 574$$

$$\sigma_{AB} = \sigma_{f(x)\varepsilon_1} = \sqrt{[(574)(0.2)]^2 + [(207.62)(1)]^2 + [(207.62)(0.2)]^2} = \sqrt{[114.8]^2 + [207.62]^2 + [41.52]^2} = \sqrt{13179.04 + 43106.06 + 1724.24} = 240.85$$

This result is shown in Table 7-2. Completing these operations for all nine WBS elements results in the other figures provided in this table. Note, the mean does not change between $\mu_{f(x)_i}$ and $\mu_{f(x)\varepsilon_i}$, but the standard deviation $\sigma_{f(x)\varepsilon_i}$ is greater than $\sigma_{f(x)_i}$ due to the effects of the estimating error, σ_{ε_i}. Now that we have nine WBS elements expressed as random variables with means and sigmas defined, we can use the FRISK method to statistically sum them. Remember from Table 4-5, $\mu_{Total} = \sum_{i=1}^{n} \mu_{f(x)_i} = 1756$. We will assume a single value for the inter-element correlations, $\rho = 0.2$, to calculate the total cost sigma,

$$\sigma_{Total} = \sqrt{\sum_{k=1}^{n}\left(\sigma_{f(x)_k}\right)^2 + 2\rho \sum_{j>i}\sum_{i=1}^{n} \sigma_{f(x)\varepsilon_i}\sigma_{f(x)\varepsilon_j}} = 476.34.$$

Table 7-2 Propagation of Errors Example Solution

WBS Element, i	$\mu_{f(x)_i}$	$\sigma_{f(x)_i}$	σ_{ε_i}	$\mu_{f(x)\varepsilon_i}$	$\sigma_{f(x)\varepsilon_i}$
Antenna	574	207.62	0.20	574	240.85
Electronics	290	105.08	0.31	290	142.07
Platform	84	22.63	0.40	84	41.51
Facilities	18	3.67	0.20	18	5.20
Power Distribution	232	83.86	0.35	232	120.37
Computers	58	11.43	0.30	58	21.10
Environmental Control	33	11.88	0.30	33	15.87
Communications	120	25.31	0.30	120	44.66
Software	347	123.68	0.30	347	165.86
TOTAL (not necessarily the sum)	1756	364.93		1756	476.34

8 Functional Correlation between WBS Elements

In Section 3.4.2 we stated that correlation can be induced by the functional relationships among random variables in an estimating model such as a schedule network or a series of cost estimating relationships. By definition, when an estimating relationship such as $Y = aX_1{}^bX_2{}^c\varepsilon$ contains a random variable, its probability distribution (Y, a dependent random variable) is dependent on the probability distributions of its inputs, X_i, (the independent random variables) and the estimating error, ε. If the dependent variable (Y) is a positive function[49] of the independent variables (i.e., $Y = aX_1{}^bX_2{}^c\varepsilon$), then the independent and dependent variables will be positively correlated (i.e., $0 < \rho_{Y,X_i} \leq 1$). Likewise, if Y is a negative function of an independent variable, they will be negatively correlated (i.e., $-1 \leq \rho_{Y,X_i} < 0$). This type of correlation is called "functional correlation" (Coleman & Gupta, 1994). There are many types of functional correlations, and if we are to use MOM techniques to estimate the probabilistic costs of multiple WBS elements (Table 8-1), it requires we have knowledge of these correlations. In this example, which pertains to the first three CERs in Table 8-1, we are interested in the correlation between Y and its independent variables, ρ_{Y,X_i}.

Table 8-1 Functional Correlation Example Cost Model

i	WBS Element, i	CER, i	Drivers	X_i	ε_i
1	Systems Engineering, Program Management Integration and Test	$Y_1 = 0.498X_1{}^{0.9}\varepsilon_1$	PMP	$\sim L\left(\sum_{i=2}^{10}\mu_i\ ,\ \sqrt{\sigma^T\rho\sigma}\right)$	L(1,0.49)
	Prime Mission Product (PMP)	$\sum_{i=2}^{10} Y_i$	Sum of Hardware and Software costs		0
2	Antenna	$Y_2 = 34.36X_{2a}{}^{0.5}X_{2b}{}^{0.8}\varepsilon_2$	Aperture Diameter (m), Frequency (GHz)	T(2,3,4) T(16,17,18)	L(1,0.30)
3	Electronics	$Y_3 = 30.06X_3{}^{0.8}\varepsilon_3$	Frequency (GHz)	T(16,17,18)	L(1,0.40)
4	Platform	$Y_4 = 26.91X_{4a}{}^{0.5}X_{4b}{}^{0.85}\varepsilon_4$	Aperture Diameter (m), Number of Axes	T(2,3,4) Constant = 2	L(1,0.38)
5	Facilities	$Y_5 = 1.64X_5{}^{0.8}\varepsilon_5$	Area (m^2)	T(18,20,22)	L(1,0.25)
6	Power Distribution	$Y_6 = 0.32X_6{}^{0.9}\varepsilon_6$	Electrical Power (W)	T(1200,1425,1875)	L(1,0.18)
7	Computers	$Y_7 = 0.58X_7{}^{0.87}\varepsilon_7$	MFLOPS	T(180,200,220)	L(1,0.31)
8	Environmental Control	$Y_8 = 1.94X_8{}^{0.4}\varepsilon_8$	Heat Load (W)	T(1100,1200,1300)	L(1,0.21)
9	Communications	$Y_9 = 5.62X_9{}^{0.9}\varepsilon_9$	Data Rate (MBPS)	T(25,30,35)	L(1,0.28)
10	Software	$Y_{10} = 1.38X_{10}{}^{1.2}\varepsilon_{10}$	eKSLOC	T(80,90,130)	L(1,0.32)

Also, if two CERs are dependent on the same random variable, X, (such as CERs 2 and 3), then those CERs will be functionally correlated to each other. Also, the common driver

[49] A positive function is one where Y increases with X.

will be correlated to those CERs. We will need to know these correlations, particularly since these variables are to be statistically summed.

Another case that is easy to envision is where one CER is a function using the sum of multiple WBS elements as its cost driver (i.e., CER 1 in Table 8-1).[50] We often refer to these types of CERs as "cost-on-cost" functions since the cost of one WBS element is a function of the cost of other WBS elements (for example, a CER that estimates program management costs and is dependent on the sum of hardware and software prime mission product (PMP) costs). In this case, we will be interested in the correlation between the cost-on-cost CER and each of the individual PMP costs.

These correlations are further complicated when correlated uncertainty terms are used in a set of CERs (e.g., $Y_2 = f_2(X)\varepsilon_2$ and $Y_3 = f_3(X)\varepsilon_3$). This is a very complex type of functional correlation since there are two dependencies involved.

Each of these cases involves a calculation of the correlation between different types of relationships between random variables. We require a more formalized approach to identifying types of functional correlations that exist in the WBS structure, or for that matter a schedule network, and how directly the random variables are related to each other. No less important is the "order", or how closely related two functionally correlated random variables are to each other. In a first order relationship, Y is clearly identified as a function of X, such as in a CER. In a second order relationship, Y may be a function of $g(X)$ (i.e., the sum of multiple random variables), one of which may be X. The third type of relationship is one in which two variables are correlated through functional relationships of other variables that are correlated. Table 8-2 provides a framework for identifying the type and order of functional correlations based on the mathematical solution to calculating ρ.

Table 8-2 Formalized Types and Orders of Functional Correlations

	Order 1	**Order 2**
Type I	$\rho_{X,Y}$ where $Y = f(X)$	$\rho_{X,Y}$ where $Y = f(g(X))$
Type II	ρ_{Y_1,Y_2} where $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$	ρ_{Y_1,Y_2} where $Y_1 = f_1\big(g_1(X)\big)$ and $Y_2 = f_2(g_2(X))$
Type III	ρ_{Y_1,Y_2} where $Y_1 = f_1(X_1)\varepsilon_1$, $Y_2 = f_2(X_2)\varepsilon_2$, and $\rho_{\varepsilon_1,\varepsilon_2} \neq 0$ or $\rho_{X_1,X_2} \neq 0$	ρ_{Y_1,Y_2} where $Y_1 = f_1(g_1(X_1)\varepsilon_1)$, $Y_2 = f_2(g_2(X_2)\varepsilon_2)$, and $\rho_{\varepsilon_1,\varepsilon_2} \neq 0$, or $\rho_{X_1,X_2} \neq 0$

With the aid of this formalized framework for segregating the types of functional correlations existing in an estimate, we can employ an organized method to find the

[50] CER 1 in the example model shown in Table 8-1.

equations for the functional correlation for each type and order described above. The method of calculating first order correlation coefficients contains the following steps:

1) Equate the correlation between two random variables in terms of Equation 4-26.
2) Determine the components of Equation 4-26.
    a. Find the means of the two RVs
    b. Find the variances of the two RVs
    c. Find the product of the two RVs
    d. Find the expectation of 2c
3) Rewrite Equation 4-26 in terms of the components found in Steps 2a through 2d.

Second order correlation coefficients require an intermediate step whereby $g(X)$ must be calculated, followed by the calculations of $\rho_{X,g(X)}$ and $\rho_{Y,g(X)}$ for Type I correlations, $\rho_{Y_1,g(X)}$ and $\rho_{Y_2,g(X)}$ for Type II correlations, and $\rho_{Y_1,g(X)\varepsilon_1}$ and $\rho_{Y_2,g(X)\varepsilon_2}$ for Type III correlations.

8.1 Type I-1 Functional Correlation

In cost analysis applications, we are often faced with the problem of computing the Type I-1 functional correlation between a CER and one of its drivers. We discussed this case when introducing functional correlation, so we will provide a method of calculating $\rho_{X_1,Y}$, where $Y = aX_1{}^bX_2{}^c\varepsilon$.

Following the process described above, Step 1: $\rho_{X_1,Y} = \frac{\mathrm{E}[X_1Y]-\mathrm{E}[X_1]\mathrm{E}[Y]}{\sqrt{Var(X_1)}\sqrt{Var(Y)}}$

Step 2a: $\mathrm{E}[X_1] = \mu_{X_1}$, which is known since X_1 is a user-defined distribution

$\mathrm{E}[Y] = \mathrm{E}[f(X_1, X_2)] = \mu_f$, which can be found through expectation methods or through the use of Mellin transforms

Step 2b: $Var(X_1)$ is known since X_1 is a user-defined distribution

$Var(Y) = \left(\mu_f\sigma_\varepsilon\right)^2 + \left(\sigma_f\right)^2 + \left(\sigma_f\sigma_\varepsilon\right)^2$; where
σ_ε is known by definition
μ_f was found in Step 2a
σ_f can be found through expectation methods or through the use of Mellin transforms

Step 2c: $X_1Y = (X_1)\left(aX_1{}^bX_2{}^c\varepsilon\right) = aX_1{}^{b+1}X_2{}^c\varepsilon$

Step 2d: $E[X_1Y] = E\left[aX_1{}^{b+1}X_2{}^c\varepsilon\right]$

Since a is a constant and the terms $X_1{}^{b+1}$, $X_2{}^c$ and ε are independent, then

$E[X_1 Y] = aE\left[{X_1}^{b+1}\right]E[{X_2}^c]E[\varepsilon]$

If we can assume $E[\varepsilon] = \mu_\varepsilon = 1$, then $E[X_1 Y] = aE\left[{X_1}^{b+1}\right]E\left[{X_2}^c\right]$.

The k^{th} moment of a RV of a known distribution type (i.e., $E[X^k]$ where X is a uniform, triangular, normal or lognormal distribution) can be calculated using Mellin transforms or through expectation operations found in Appendix B – Expectation Operations.

Step 3: Combining the terms from steps 1 through 2d we have

$$\rho_{X_1,Y} = \frac{aE[{X_2}^c]\left(E\left[{X_1}^{b+1}\right] - E\left[{X_1}^b\right]E[X_1]\right)}{\sigma_{X_1}\sqrt{(\mu_f \sigma_\varepsilon)^2 + (\sigma_f)^2 + (\sigma_f \sigma_\varepsilon)^2}} \qquad \textbf{8-1}$$

Equation 8-1 shows that as the magnitude of σ_ε increases, the magnitude of $\rho_{X_1,Y}$ decreases.

8.1.1 Type I-1 Functional Correlation Example

For this example, we will use CER 6 from Table 8-1 to calculate the Type I-1 functional correlation between Y_6 and its driver, X_6. The CER Y_6 is defined as

$$Y_6 = 0.32{X_6}^{0.9}\varepsilon_6$$

Following the process described above, Step 1: $\rho_{X_6,Y_6} = \frac{\mathrm{E}[X_6 Y_6] - \mathrm{E}[X_6]\mathrm{E}[Y_6]}{\sqrt{Var(X_6)}\sqrt{Var(Y_6)}}$

Step 2a: $\mathrm{E}[X_6] = \mu_{X_6}$, which is found using Equation 4-1.

Since X_6 is defined by the triangular PDF, $T(1200,1425,1875)$,

$\mathrm{E}[X_6] = \mu_{X_6} = \frac{1200+1425+1875}{3} = 1500$

$\mathrm{E}[Y_6]$ can be found through expectation methods or through the use of Mellin transforms. In this example, we will use expectation methods to compute $\mathrm{E}[Y_6]$.

$\mathrm{E}[Y_6] = \mathrm{E}\left[0.32{X_6}^{0.9}\varepsilon_6\right] = 0.32\mathrm{E}\left[{X_6}^{0.9}\right]\mathrm{E}[\varepsilon_6]$, and since $\mathrm{E}[\varepsilon_6] = 1$, $\mathrm{E}[Y_6] = 0.32\mathrm{E}\left[{X_6}^{0.9}\right]$.

Since X_6 is a triangular PDF, we must find the expectation of a triangular PDF raised to a power, which is

$$E\left[X^{\mathrm{k}}\right] = \frac{2}{(H-L)(M-L)}\left\{\frac{M^{k+2}-L^{k+2}}{k+2} - L\frac{M^{k+1}-L^{k+1}}{k+1}\right\} + \frac{2}{(H-L)(H-M)}\left\{H\frac{H^{k+1}-M^{k+1}}{k+1} - \frac{H^{k+2}-M^{k+2}}{k+2}\right\}$$

Substituting the parameters L, M, H and k using our example, $\mathrm{E}\left[{X_6}^{0.9}\right] = 721.626$

So $\mathrm{E}[Y_6] = (0.32)(721.626) = 230.920$.

Step 2b: $Var(X_6)$ is calculated using the square of one half of the population standard deviation of the distributions parameters. This equates to

$Var(X_6) = \left(\frac{STDEVP(1200,1425,1875)}{2}\right)^2 = 19687.5$, so $\sigma_{X_6} = \sqrt{19687.5} = 140.31$

The variance of Y is calculated using the propagation of errors method, since the CER, f_{Y_6}, and its error are independent RVs.

$Var(Y) = \left(\mu_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2 + \left(\sigma_{f_{Y_6}}\right)^2 + \left(\sigma_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2$; where

$\sigma_{\varepsilon_6} = 0.18$ (Table 8-1), and $\mu_{f_{Y_6}} = 230.920$ (found in Step 2a)

$\sigma_{f_{Y_6}}$ can be found through expectation methods or through the use of Mellin transforms. In this case, we will use the equation for the transformation of a triangular PDF from Section 4.3.3 to compute this value.

$$\sigma_{f_{Y_6}} = b\sqrt{\frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{2c+2}-L^{2c+2}}{2c+2} - L\frac{M^{2c+1}-L^{2c+1}}{2c+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{2c+1}-M^{2c+1}}{2c+1} - \frac{H^{2c+2}-M^{2c+2}}{2c+2}\right\}\right] - \left(\frac{\mu_f}{b}\right)^2}$$

By substituting the coefficient $b = 0.32$ and the triangular distribution parameters, L, M and H into this equation, we get $\sigma_{f_{Y_6}} = 19.428$.

So $\sigma_{Y_6} = \sqrt{\left(\mu_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2 + \left(\sigma_{f_{Y_6}}\right)^2 + \left(\sigma_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2}$

$\sigma_{Y_6} = \sqrt{[(230.920)(0.18)]^2 + [19.428]^2 + [(19.428)(0.18)]^2} = 46.015$

In Step 2c we calculate the product X_6Y_6 through expansion.

$X_6Y_6 = (X_6)\left(0.32{X_6}^{0.9}\varepsilon_6\right) = 0.32{X_6}^{1.9}\varepsilon_6$

In Step 2d we calculate the expectation of this product.

$E[X_6Y_6] = E\left[0.32{X_6}^{1.9}\varepsilon_6\right] = 0.32E\left[{X_6}^{1.9}\right]E[\varepsilon_6]$

Since $E[\varepsilon_6] = \mu_{\varepsilon_6} = 1$, then $E[X_6Y_6] = 0.32E\left[{X_6}^{1.9}\right]$.

Using the equation for the k^{th} moment of a triangular distribution, we can compute $E[X_6Y_6]$

$E[X_6Y_6] = (0.32)(1090957.67) = 349106.45$

Furthermore, the product $E[X_6]E[Y_6] = (1500)(230.920) = 346380.516$.

Step 3: Combining the terms from Steps 1 through 2d, we have:

$$\rho_{X_6,Y_6} = \frac{E[X_6Y_6]-E[X_6]E[Y_6]}{\sigma_{X_6}\sigma_{Y_6}} = \frac{349106.45-346380.516}{(140.31)(19.428)} = 0.4222$$

8.2 Type I-2 Functional Correlation

In this case, we wish to find the functional correlation $\rho_{X,Y}$ between two random variables X_i and Y where $Y = f(g(X_i))\varepsilon_Y$. We will assume $f(W)$ is a CER, specifically a cost-on-cost function of the summation, $W = g(X_i\varepsilon_i) = \sum_{i=1}^{n} X_i$, of WBS elements where X_i is one of the summands. In this type of functional correlation, we assume W and ε_Y are independent random variables.

$Y = (a + bW^c)\varepsilon_Y$, and $W = g(X) = \sum_{i=1}^{n} X_i$

Following Step 1 of the process described above, we can express the correlation as:

$$\rho_{X,Y} = \frac{\mathrm{E}[X_iY]-\mathrm{E}[X_i]\mathrm{E}[Y]}{\sqrt{Var(X_i)}\sqrt{Var(Y)}} = \frac{\mathrm{E}[X_if(g(X_i))]-\mathrm{E}[X_i]\mathrm{E}[f(g(X_i))]}{\sqrt{Var(X_i)}\sqrt{Var(f(g(X_i)))}}$$

Rewriting these terms, $\rho_{X,Y} = \dfrac{\mathrm{E}\left[X_i\left(a+b\left[\sum_{i=1}^{n} X_i\right]^c\right)\varepsilon_Y\right]-\mathrm{E}[X_i]\mathrm{E}\left[\left(a+b\left[\sum_{i=1}^{n} X_i\right]^c\right)\varepsilon_Y\right]}{\sigma_{X_i}\sqrt{Var\left[\left(a+b\left[\sum_{i=1}^{n} X_i\right]^c\right)\varepsilon\right]}}$

In Step 2a, we must find the means of X_i and Y.

$\mathrm{E}[X_i] = \mu_{X_i}$, which is known since X_i is a WBS element summand and can be calculated using either expectation methods or through Mellin transforms.

$\mathrm{E}[Y] = \mathrm{E}[(a + b[\sum_{i=1}^{n} X_i]^c)\varepsilon_Y] = \mathrm{E}[a\varepsilon + b[\sum_{i=1}^{n} X_i]^c\varepsilon_Y]$

This expression can be rewritten as:

$\mathrm{E}[Y] = a\mathrm{E}[\varepsilon] + b\mathrm{E}[(\sum_{i=1}^{n} X_i)^c\varepsilon_Y]\mathrm{E}[\varepsilon_Y] = a + b\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]$, since $\mathrm{E}[\varepsilon_Y] = 1$

$\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]$ can be found for a lognormal PDF since $E\left[X^{\mathrm{k}}\right] = e^{\left(kP+\frac{1}{2}Q^2k^2\right)}$

In Step 2b, we find the variances of X_i and Y.

$Var(X_i)$ is assumed to be known, and $Var(Y)$ is calculated using the propagation of errors method.

In Step 2c, we find the product X_iY through expansion.

$X_iY = X_i(a + bW^c)\varepsilon_Y = aX_i\varepsilon_Y + bX_iW^c\varepsilon_Y = aX_i\varepsilon_Y + b\varepsilon_YX_i(\sum_{i=1}^{n} X_i)^c$

We must move $X_i$into the summation, $\sum_{i=1}^{n} X_i\, \varepsilon_i$, which results in:

$$X_i Y = aX_i\varepsilon_Y + b\varepsilon_Y\left(\sum_{i=1}^{n} X_i^{\frac{1}{c}} X_i\right)^c = aX_i\varepsilon_Y + b\varepsilon_Y\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c$$

Now we have separable terms from which to compute the expectation.

In Step 2d, the expectation is $\mathrm{E}[X_i Y] = aX_i\varepsilon_Y + b\varepsilon_Y\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c$. In the next step, we face a conundrum. We already assume that ε_Y and W are independent RVs as a condition of the regression of the CER, $f(W)$. We may also assume X_i contains some multiplicative error, ε_i, so that that error must be independent of $f(W)$and ε_Y. In practice, however, this case is not always true, since sample correlations do exist between ε_i and ε_Y. We must assume that independence overrides this situation and that X_i, ε_Y and ε_i are all independent RVs. Given this, the expectation can be reduced to:

$\mathrm{E}[X_i Y] = aE[X_i]E[\varepsilon_Y] + bE[\varepsilon_Y]E\left[\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c\right]$, and since $E[\varepsilon_Y] = 1$,

$\mathrm{E}[X_i Y] = a\mu_{X_i} + b\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c\right]$, which is solvable knowing $\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)$ is lognormally distributed and that $E\left[X^{\mathrm{k}}\right] = e^{\left(kP+\frac{1}{2}Q^2k^2\right)}$.

Since $\mathrm{E}[X_i] = \mu_{X_i}$, and $\mathrm{E}[Y] = a + b\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]$, the product of the expectations of X_i and Y is $\mathrm{E}[X_i]\mathrm{E}[Y] = \mu_{X_i}(a + b\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]) = a\mu_{X_i} + b\mu_{X_i}\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]$

The term $\mathrm{E}[X_i Y] - \mathrm{E}[X_i]\mathrm{E}[Y]$ is reduced to

$$\mathrm{E}[X_i Y] - \mathrm{E}[X_i]\mathrm{E}[Y] = a\mu_{X_i} + b\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c\right] - a\mu_{X_i} - b\mu_{X_i}\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]$$

$$\mathrm{E}[X_i Y] - \mathrm{E}[X_i]\mathrm{E}[Y] = b\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c\right] - \mu_{X_i}\mathrm{E}[(\sum_{i=1}^{n} X_i)^c]\right\}$$

In step 3, we find the functional correlation $\rho_{X,Y}$ by combining terms into the expression found in Step 1.

$$\rho_{X,Y} = \frac{b\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i^{\left[1+\frac{1}{c}\right]}\right)^c\right] - \mu_{X_i}\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^c\right]\right\}}{\sqrt{Var(X_i)}\sqrt{Var(Y)}}$$

8.2.1 Type I-2 Functional Correlation Example

In this example, we show how to find the functional correlation between CERs 1 and 2, ρ_{Y_1,Y_2}, in our example model. CER 1 is a cost-on-cost function of the summation of WBS elements 2 through 10 (i.e., $W = \sum_{i=2}^{10} Y_i$) , where the cost of WBS element 2 (i.e., Y_2) is one of the summands. The CERs are:

$Y_1 = (0.498(\sum_{i=2}^{10} Y_i)^{0.9})\varepsilon_1$, and $Y_2 = \left(34.36{X_{2a}}^{0.5}{X_{2b}}^{0.8}\right)\varepsilon_2$, where ε_1 and $\varepsilon_2$are multiplicative errors of the CERs defined by $L(1{,}0.45)$ and $L(1{,}0.3)$, respectively.

In this type of functional correlation, we assume W (the sum of Y_i) and ε_1 are independent lognormal RVs. Following Step 1 of the process described above, we can express the correlation between CERs 1 and 2 as:

$$\rho_{Y_1,Y_2} = \frac{\mathrm{E}[Y_1Y_2]-\mathrm{E}[Y_1]\mathrm{E}[Y_2]}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}}$$

Substituting the functional forms of CERs 1 and 2 into these terms results in:

$$\rho_{Y_1,Y_2} = \frac{\mathrm{E}\left[Y_2\left\{b\left(\sum_{i=2}^{10} W\right)^c\right\}\varepsilon_{Y_1}\right]-\mathrm{E}[Y_2]\mathrm{E}\left[\left\{b\left(\sum_{i=2}^{10} W\right)^c\right\}\varepsilon_{Y_1}\right]}{\sigma_{Y_1}\sigma_{Y_2}}$$

8.2.1.1 Means of Correlated Random Variables

In Step 2a, we find $E[Y_1]$ and $E[Y_2]$, which are the means of WBS elements 1 and 2.

$\mathrm{E}[Y_2] = \mu_{Y_2}$, which is calculated using expectation methods, is

$$\mathrm{E}[Y_2] = E\left[\left(34.36{X_{2a}}^{0.5}{X_{2b}}^{0.8}\right)\varepsilon_2\right] = 34.36E\left[{X_{2a}}^{0.5}\right]E\left[{X_{2b}}^{0.8}\right]$$

From the previous example, we calculated $\mathrm{E}[Y_2]$ using the product of k^{th} expectation of the triangularly distributed independent variables $X_{2a}$and X_{2b}. The result is repeated here.

$$\mathrm{E}[Y_2] = (34.36)(1.728)(9.646) = 572.706$$

Using this method for the remaining CERs in WBS elements 3 to 10 by substituting their respective PDFs and CER coefficients, we can calculate their means. We sum the means of CERs 2 through 10 to get the mean of their sum, since $\mathrm{E}[\sum Y_i] = \sum E[Y_i]$. These results are shown in Table 8-3.

Table 8-3 Means of CERs of WBS Elements 2 through 10

$\mathbf{CER}_i$	b_i	$\mu_{x_{ia}}$	$\mu_{x_{ib}}$	μ_{Y_i}
2	34.360	1.728	9.646	572.706
3	30.060	9.646	-	289.953
4	26.910	1.728	1.803	83.816
5	1.640	10.984	-	18.014
6	0.320	721.626	-	230.920
7	0.580	100.428	-	58.248
8	1.940	17.046	-	33.068
9	5.620	21.346	-	119.965
10	1.380	251.536	-	347.120
SUM	-	-	-	1753.813

The mean of CER 1 is defined as $\mathrm{E}[Y_1] = \mathrm{E}\left[\left(b\left(\sum_{i=2}^{10} Y_i\right)^c\right)\varepsilon_{Y_1}\right] = b\mathrm{E}\left[\left(W^c\right)\varepsilon_{Y_1}\right]$, where W is the RV of the sum of WBS elements 2 through 10.

This expression can be rewritten as $\mathrm{E}[Y_1] = b\mathrm{E}[W^c]\mathrm{E}\left[\varepsilon_{Y_1}\right] = b\mathrm{E}[W^c]$, since $\mathrm{E}\left[\varepsilon_{Y_1}\right] = 1$.

We can also assume that the sum, W, represents a lognormal distribution with the parameters P_W and Q_W that define W's underlying normal distribution. P_W and Q_W are dependent on both the mean and variance of W (i.e., μ_W and $\sigma_W{}^2$).

The term $\mathrm{E}[W^c]$ can be found for a lognormal PDF since $E[W^c] = e^{\left(cP_W + \frac{1}{2}Q_W{}^2 c^2\right)}$, but P_W and Q_W are functions of μ_W and σ_W. We must complete Step 2b in order to compute the values of the following: σ_{Y_i}, for each Y_i; μ_W and σ_W; P_W and Q_W; $E[W^c]$ and σ_{W^c}; and, finally $\mathrm{E}[Y_1]$ and σ_{Y_1}.

8.2.1.2 Standard Deviations of Correlated Random Variables

Each σ_{Y_i} for CERs 2 through 10 is calculated using the propagation of errors method. They are reported as σ_{Y_i} in Table 8-4.

Table 8-4 Means and Standard Deviations of CERs of WBS Elements 2 through 10

CER i	b_i	$\mu_{x_{ia}}$	$\mu_{x_{ib}}$	μ_{Y_i}	σ_{ϵ_i}	$\sigma_{x_{ia}}$	$\sigma_{x_{ib}}$	$\sigma_{f(x_i)}$	σ_{Y_i}
2	34.360	1.728	9.646	572.706	0.3	0.1186	0.1853	40.8333	177.0219
3	30.060	9.646	-	289.953	0.4	0.1853	-	5.5711	116.1364
4	26.910	1.728	1.803	83.816	0.38	0.1186	0.0001	5.7539	32.4396
5	1.640	10.984	-	18.014	0.25	0.3589	-	0.5885	4.5442
6	0.320	721.626	-	230.920	0.18	60.7123	-	19.4279	46.0150
7	0.580	100.428	-	58.248	0.31	3.5677	-	2.0692	18.1865
8	1.940	17.046	-	33.068	0.21	0.2321	-	0.4503	6.9596
9	5.620	21.346	-	119.965	0.28	1.3077	-	7.3494	34.4464
10	1.380	251.536	-	347.120	0.32	32.7041	-	45.1317	120.7638
W	-	-	-	1753.813	-	-	-	-	331.911

We find μ_W in Table 8-3. The standard deviation of W is found through linear algebra using the relationship $\sigma_W = \sqrt{\boldsymbol{\sigma_Y}^{\boldsymbol{T}} \boldsymbol{\rho_Y} \boldsymbol{\sigma_Y}}$. In this relationship, $\boldsymbol{\sigma_Y}$ is the vector of $\sigma_{Y_i}$for $2 \leq i \leq 10$, $\boldsymbol{\sigma_Y}^{\boldsymbol{T}}$ is the transpose of that vector, and $\boldsymbol{\rho_Y}$ is the functional correlation between CERs of WBS elements 2 through 10. The matrix $\boldsymbol{\rho_Y}$ is a 9x9 element sub-matrix of the entire 10x10 functional correlation matrix. In this case, we need the lower 9 rows and columns to calculate the first row and first column of the full 10x10 matrix.

In our example, all elements of $\boldsymbol{\rho_Y}$ are Type III-1 or Type II-1 functional correlations, for which we provide examples in other parts of this section.

$$\boldsymbol{\rho_Y} = \begin{bmatrix} 1 & 0.1969 & 0.2309 & 0.1924 & 0.1753 & 0.1927 & 0.1937 & 0.1893 & 0.1785 \\ 0.1969 & 1 & 0.1961 & 0.1979 & 0.1804 & 0.1983 & 0.1993 & 0.1948 & 0.1837 \\ 0.2309 & 0.1961 & 1 & 0.1946 & 0.1774 & 0.1950 & 0.1959 & 0.1915 & 0.1806 \\ 0.1924 & 0.1979 & 0.1946 & 1 & 0.1790 & 0.1968 & 0.1978 & 0.1933 & 0.1823 \\ 0.1753 & 0.1804 & 0.1774 & 0.1790 & 1 & 0.1794 & 0.1803 & 0.1762 & 0.1662 \\ 0.1927 & 0.1983 & 0.1950 & 0.1968 & 0.1794 & 1 & 0.1981 & 0.1936 & 0.1827 \\ 0.1937 & 0.1993 & 0.1959 & 0.1978 & 0.1803 & 0.1981 & 1 & 0.1946 & 0.1836 \\ 0.1893 & 0.1948 & 0.1915 & 0.1933 & 0.1762 & 0.1936 & 0.1946 & 1 & 0.1794 \\ 0.1785 & 0.1837 & 0.1806 & 0.1823 & 0.1662 & 0.1827 & 0.1836 & 0.1794 & 1 \end{bmatrix}$$

Knowing the values of the 1x9 vector $\boldsymbol{\sigma_Y}$ and the 9x9 matrix $\boldsymbol{\rho_Y}$, the standard deviation of W is calculated through the linear algebraic relationship $\sigma_W = \sqrt{\boldsymbol{\sigma_Y}^{\boldsymbol{T}}\boldsymbol{\rho_Y}\boldsymbol{\sigma_Y}} = 331.911.$

Using $\mu_W = 1753.813$ and $\sigma_W = 331.911$, we can calculate P_W and Q_W, where:

$P_W = \frac{1}{2} ln\left(\frac{\mu_W{}^4}{\mu_W{}^2 + \sigma_W{}^2}\right) = 7.452$, and $Q_W = \sqrt{ln\left(1 + \frac{\sigma_W{}^2}{\mu_W{}^2}\right)} = 0.188.$

Now that the parameters of the underlying normal distribution of W are known, we can calculate values of $E[W^c]$ and subsequently $E[Y_1]$ and σ_{Y_1}.

First, $E[W^c] = e^{\left(cP + \frac{1}{2}Q^2c^2\right)} = e^{\left((0.9)(7.452) + \frac{1}{2}\left((0.9)(0.188)\right)^2\right)} = 829.654$, and since $E[Y_1] = bE[W^c]$, then

$E[Y_1] = \mu_{Y_1} = \mu_{f_{X_1}} = (0.498)(829.654) = 413.168.$

We can express Y_1 as $Y_1 = (b(\sum_{i=2}^{10} Y_i)^c)\varepsilon_{Y_1} = (bW^c)\varepsilon_{Y_1} = f_{W_1}\varepsilon_{Y_1}$. Since we need to find σ_{Y_1}, and it is formed by the product of f_{W_1} and its multiplicative error, we must first find $\sigma_{f_{W_1}}$ then account for the multiplicative error. Since W is exponentiated by the coefficient, c, we must calculate the standard deviation of f_{W_1} using W's underlying normal distribution (defined by P_W and Q_W), then find the log transformation of the scaled normal distribution. From this process, we obtain:

$\sigma_{f_{W_1}} = b\sqrt{e^{\left(2cP_W + \frac{1}{2}[cQ_W]^2\right)}\left(e^{[cQ_W]^2} - 1\right)} = (0.498)\sqrt{e^{(2(0.9)(7.452)+(0.5)[(0.9)(0.188)]^2)}\left(e^{[(0.9)(0.188)]^2} - 1\right)}$, so

$\sigma_{f_{W_1}} = 69.756.$

Using the propagation of errors method, we can compute σ_{Y_1} knowing $\mu_{f_{W_1}}, \sigma_{f_{W_1}}$, and σ_{ε_1}.

$$\sigma_{Y_1} = \sqrt{\left(\mu_{f_{W_1}}\sigma_{\varepsilon_1}\right)^2 + \left(\sigma_{f_{W_1}}\right)^2 + \left(\sigma_{f_{W_1}}\sigma_{\varepsilon_1}\right)^2} = 201.046.$$

8.2.1.3 Expectation of Product of Correlated Random Variables

Our work is not complete since we still need to calculate the numerator of the correlation equation in Step 1.

In Step 2c, we find the product Y_1Y_2 to be $Y_1Y_2 = Y_2(b(\sum_{i=2}^{10} Y_i)^c)\varepsilon_1$.

Moving the RV Y_2 into the summation results in $Y_1Y_2 = b\left(\sum_{i=2}^{10} Y_iY_2^{\frac{1}{c}}\right)^c \varepsilon_1$.

In Step 2d, the expectation of the product Y_1Y_2 is $\mathrm{E}[Y_1Y_2] = E\left[b\left(\sum_{i=2}^{10} Y_iY_2^{\frac{1}{c}}\right)^c \varepsilon_1\right]$, which reduces to $\mathrm{E}[Y_1Y_2] = bE[\varepsilon_1]E\left[\left(\sum_{i=2}^{10} Y_iY_2^{\frac{1}{c}}\right)^c\right]$.

Since $E[\varepsilon_1] = 1$, we can further reduce this to $\mathrm{E}[Y_1Y_2] = bE\left[\left(\sum_{i=2}^{10} Y_iY_2^{\frac{1}{c}}\right)^c\right] = bE[V^c]$.

This is solvable knowing the following: the means and variances of the products, $V_i = Y_iY_2^{\frac{1}{c}}$, are calculable; the products can be summed to form the random variable, V, where $V = \sum V_i$; and the term V is lognormally distributed, so $E[V^c] = e^{\left(cP_U + \frac{1}{2}Q_U^2c^2\right)}$.

We start with calculating the moments of the product $Y_iY_2^{\frac{1}{c}}$. As an example, we will set $i = 3$ and find the mean and variance of $V_3 = Y_3Y_2^{\frac{1}{c}}$. Using the method described in Section 5.3, we define the lognormal RVs, Y_2 and Y_3, using the normally distributed RVs, Z_2 and Z_3.

$$V_3 = Y_3Y_2^{\frac{1}{c}} = e^{(Z_3 + Z_2/c)}$$

$P_U = P_{Z_3} + \frac{1}{c}P_{Z_2}$ and $Q_U^2 = Q_{Z_3}^2 + 2\rho_{Z_2,Z_3}\frac{1}{c}Q_{Z_2}Q_{Z_3} + \frac{1}{c^2}Q_{Z_2}^2$, where

$$\rho_{Z_2,Z_3} = \frac{1}{Q_{Z_2}Q_{Z_3}}\ln\left[1 + \rho_{Y_2,Y_3}\left(\sqrt{e^{Q_{Z_2}^2} - 1}\sqrt{e^{Q_{Z_3}^2} - 1}\right)\right].$$

Using Equations 4-5 and 4-6 with values for μ_{Y_2}, σ_{Y_2}, μ_{Y_3}, and σ_{Y_3} from Table 8-4, we obtain:

$$P_{Z_2} = \frac{1}{2}ln\left(\frac{{\mu_{Y_2}}^4}{{\mu_{Y_2}}^2 + {\sigma_{Y_2}}^2}\right) = \frac{1}{2}ln\left(\frac{(572.706)^4}{(572.706)^2 + (177.022)^2}\right) = 6.305,$$

$$Q_{Z_2} = \sqrt{ln\left(1 + \frac{{\sigma_{Y_2}}^2}{{\mu_{Y_2}}^2}\right)} = \sqrt{ln\left(1 + \frac{(177.022)^2}{(572.706)^2}\right)} = 0.302,$$

$$P_{Z_3} = \frac{1}{2}ln\left(\frac{{\mu_{Y_3}}^4}{{\mu_{Y_3}}^2 + {\sigma_{Y_3}}^2}\right) = \frac{1}{2}ln\left(\frac{(289.953)^4}{(289.953)^2 + (116.136)^2}\right) = 5.595,$$

$Q_{Z_3} = \sqrt{ln\left(1 + \frac{\sigma_{Y_3}{}^2}{\mu_{Y_3}{}^2}\right)} = \sqrt{ln\left(1 + \frac{(116.136)^2}{(289.953)^2}\right)} = 0.386$, and the correlation between this pair of normal RVs is calculated as:

$$\rho_{Z_2,Z_3} = \frac{1}{Q_{Z_2}Q_{Z_3}} \ln\left[1 + \rho_{Y_2,Y_3}\left(\sqrt{e^{Q_{Z_2}^2} - 1}\sqrt{e^{Q_{Z_3}^2} - 1}\right)\right]$$

$$\rho_{Z_2,Z_3} = \frac{1}{(0.302\ \ )(0.386)} \ln\left[1 + (0.1961)\left(\sqrt{e^{(0.302\ )^2} - 1}\sqrt{e^{(0.386)^2} - 1}\right)\right] = 0.2067$$

So the new distribution formed by the product $Y_3 Y_2^{\frac{1}{c}}$ has an underlying normal distribution, U_3, where:
$P_{U_3} = P_{Z_3} + \frac{1}{c}P_{Z_2} = 5.595 + \frac{1}{0.9}6.305 = 12.601$, and
$Q_{U_3}^2 = Q_{Z_3}^2 + 2\rho_{Z_2,Z_3}\frac{1}{c}Q_{Z_2}Q_{Z_3} + \frac{1}{c^2}Q_{Z_2}^2 = (0.386)^2 + 2(0.2067)\frac{1}{0.9}(0.386)(0.302) + \left(\frac{0.302}{0.9}\right)^2 =$
0.315

Then, the mean and variance of V_3 are found by transforming U_3 back to a lognormal distribution, V_3.

$\mu_{V_3} = e^{\left(P_{U_3} + \frac{1}{2}Q_{U_3}^2\right)} = e^{\left(12.601 + \frac{1}{2}(0.315)\right)} = 347348.652$, and

$$\sigma_{V_3} = \sqrt{e^{\left(2P_{U_3} + \frac{1}{2}Q_{U_3}^2\right)}\left(e^{Q_{U_3}^2} - 1\right)} = \sqrt{e^{(2(12.601) + (0.5)(0.315))}(e^{0.315} - 1)} = 1.953E + 05.$$

We need to repeat this procedure for all V_i, so after computing the remaining V_i terms, we obtain the results in Table 8-5.

Since V is to be exponentiated, we will need to find both its mean (μ_V) and standard deviation (σ_V) in order to perform the exponentiation. The mean of V, μ_V, is the sum of the elements μ_{V_i}, which is 2145735.39.

Table 8-5 Calculation of V_i Distribution Parameters

i	μ_{Y_i}	σ_{Y_i}	P_{Z_i}	Q_{Z_i}	ρ_{Z_2,Z_i}	P_{U_i}	Q_{U_i}	μ_{V_i}	σ_{V_i}
2	572.706	177.022	6.305	0.302	1.0000	13.310	0.638	739228.715	4.730E+05
3	289.953	116.136	5.595	0.386	0.2067	12.601	0.561	347348.652	1.953E+05
4	83.816	32.440	4.359	0.374	0.2414	11.364	0.559	100760.716	5.647E+04
5	18.014	4.544	2.860	0.248	0.1984	9.866	0.455	21360.402	9.737E+03
6	230.920	46.015	5.423	0.197	0.1802	12.428	0.419	272559.191	1.142E+05
7	58.248	18.186	4.018	0.305	0.2000	11.023	0.497	69341.165	3.448E+04
8	33.068	6.960	3.477	0.208	0.1991	10.482	0.429	39108.488	1.678E+04
9	119.965	34.446	4.748	0.281	0.1959	11.753	0.478	142530.987	6.827E+04
10	347.120	120.764	5.793	0.338	0.1863	12.798	0.519	413497.078	2.149E+05
$\sum$	1753.813	331.911	-	-	-	-	-	2145735.39	-

The standard deviation of V, σ_V, is calculated through the linear algebraic relationship, $\sigma_V = \sqrt{\boldsymbol{\sigma_V}^T \boldsymbol{\rho_V} \boldsymbol{\sigma_V}}$. To find this quantity, we need to know the values of the 9x9 correlation matrix $\boldsymbol{\rho_V}$, whose elements are $\rho_{V_i,V_j} = \rho_{Y_i Y_2{}^{1/c}, Y_j Y_2{}^{1/c}}$. This correlation matrix is formed by computing the individual 9x9 elements as follows:

$$\rho_{Y_i Y_2{}^{1/c}, Y_j Y_2{}^{1/c}} = \frac{E\left[\left(Y_i Y_2^{\frac{1}{c}}\right)\left(Y_j Y_2^{\frac{1}{c}}\right)\right] - E[V_i]E[V_j]}{\sigma_{V_i}\sigma_{V_j}} = \frac{E\left[Y_i Y_j Y_2^{\frac{2}{c}}\right] - \mu_{V_i}\mu_{V_j}}{\sigma_{V_i}\sigma_{V_j}}$$

Fortunately, we have already calculated the values of μ_{V_i} and σ_{V_i} (thus μ_{V_j} and σ_{V_j} as well) in Table 8-5, but we need to know $E\left[Y_i Y_j Y_2^{\frac{2}{c}}\right]$ in order to find $\rho_{Y_i Y_2{}^{1/c}, Y_j Y_2{}^{1/c}}$ and complete the calculation of $\sigma_V = \sqrt{\boldsymbol{\sigma_V}^T \boldsymbol{\rho_V} \boldsymbol{\sigma_V}}$.

The term $E\left[Y_i Y_j Y_2^{\frac{2}{c}}\right]$ is calculated through the triple product of lognormal RVs with one RV (Y_2) raised to a power – a task that is non-trivial but essential. Fortunately, we can solve this problem using our knowledge of the expectations of products of lognormal RVs. The triple product is formed by summing the parameters P and Q of the underlying normal distributions of Y_i, Y_j, and $Y_2^{\frac{2}{c}}$, then transforming this sum back to a lognormal distribution representing $Y_i Y_j Y_2^{\frac{2}{c}}$.

We represent the variable of the triple product of Y_i, Y_j, and $Y_2^{\frac{2}{c}}$ as a lognormal distribution, $T_{2,i,j}$, with the underlying normal distribution $S_{2,i,j}$ such that $T_{2,i,j} = e^{S_{2,i,j}}$. $S_{2,i,j}$ is defined by mean $P_{S_{2,i,j}}$ and variance $Q_{S_{2,i,j}}{}^2$ which are:

$P_{S_{2,i,j}} = P_{Z_i} + P_{Z_j} + \frac{2}{c} P_{Z_2}$, and

$$Q_{S_{2,i,j}}{}^2 =$$

$$Q_{Z_i}{}^2 + Q_{Z_j}{}^2 + \left(\frac{2}{c} Q_{Z_2}\right)^2 + 2\left\{\rho_{Z_i,Z_j} Q_{Z_i} Q_{Z_j} + \rho_{Z_i,Z_2} Q_{Z_i}\left(\frac{2}{c} Q_{Z_2}\right) + \rho_{Z_j,Z_2} Q_{Z_j}\left(\frac{2}{c} Q_{Z_2}\right)\right\},$$ where

$$\rho_{Z_i,Z_j} = \frac{1}{Q_{Z_i} Q_{Z_j}} \ln\left[1 + \rho_{Y_i,Y_j}\left(\sqrt{e^{Q_{Z_i}^2} - 1}\sqrt{e^{Q_{Z_j}^2} - 1}\right)\right]$$

For one of the elements where $i = 3$ and $j = 4$, $P_{S_{2,3,4}} = P_{Z_3} + P_{Z_4} + \frac{2}{c} P_{Z_2}$, which becomes

$$P_{S_{2,3,4}} = 5.595 + 4.359 + \frac{2}{0.9} 6.305 = 23.965.$$

The correlation coefficient of the normal distributions Z_3 and Z_4 is a transformation of ρ_{Y_3,Y_4}, which has already been calculated.

$\rho_{Z_3,Z_4} = \frac{1}{Q_{Z_3}Q_{Z_4}} \ln\left[1 + \rho_{Y_3,Y_4}\left(\sqrt{e^{Q_{Z_3}^2} - 1}\sqrt{e^{Q_{Z_4}^2} - 1}\right)\right] = \frac{1}{(0.386)(0.374)} \ln\Big[1 + (0.1961)\left(\sqrt{e^{(0.386)^2} - 1}\sqrt{e^{(0.374)^2} - 1}\right)\Big] = 0.2078$

We obtain ρ_{Z_2,Z_3} and ρ_{Z_2,Z_4} similarly.

$\rho_{Z_2,Z_3} = \frac{1}{Q_{Z_2}Q_{Z_3}} \ln\left[1 + \rho_{Y_2,Y_3}\left(\sqrt{e^{Q_{Z_2}^2} - 1}\sqrt{e^{Q_{Z_3}^2} - 1}\right)\right] = \frac{1}{(0.302)(0.386)} \ln\Big[1 + (0.1969)\left(\sqrt{e^{(0.302)^2} - 1}\sqrt{e^{(0.386)^2} - 1}\right)\Big] = 0.2067$

$\rho_{Z_2,Z_4} = \frac{1}{Q_{Z_2}Q_{Z_4}} \ln\left[1 + \rho_{Y_2,Y_4}\left(\sqrt{e^{Q_{Z_2}^2} - 1}\sqrt{e^{Q_{Z_4}^2} - 1}\right)\right] = \frac{1}{(0.302)(0.374)} \ln\Big[1 + (0.2309)\left(\sqrt{e^{(0.302)^2} - 1}\sqrt{e^{(0.374)^2} - 1}\right)\Big] = 0.2414$

Using the values of Q_{Z_2}, Q_{Z_3}, Q_{Z_4}, ρ_{Z_3,Z_4}, ρ_{Z_2,Z_3} and ρ_{Z_2,Z_4} we can get the parameters of $S_{2,3,4}$.

$Q_{S_{2,3,4}}{}^2 = {Q_{Z_3}}^2 + {Q_{Z_4}}^2 + \left(\frac{2}{c}Q_{Z_2}\right)^2 + 2\left\{\rho_{Z_3,Z_4}Q_{Z_3}Q_{Z_4} + \rho_{Z_3,Z_2}Q_{Z_3}\left(\frac{2}{c}Q_{Z_2}\right) + \rho_{Z_4,Z_2}Q_{Z_4}\left(\frac{2}{c}Q_{Z_2}\right)\right\}$

$Q_{S_{2,3,4}}{}^2 = (0.386)^2 + (0.374)^2 + \left(\frac{2(0.302)}{0.9}\right)^2 + 2\Big\{(0.2078)(0.386)(0.374) + (0.2067)(0.386)\left(\frac{2(0.302)}{0.9}\right) + (0.2414)(0.374)\left(\frac{2(0.302)}{0.9}\right)\Big\} = 1.027$

The mean of $T_{2,i,j}$ (also known as $E\left[Y_iY_j{Y_2}^{\frac{2}{c}}\right]$), is $\mu_{T_{2,i,j}} = e^{\left(P_{S_{2,i,j}} + \frac{1}{2}Q_{S_{2,i,j}}{}^2\right)}$.

So in the case where $i = 3$ and $j = 4$, $\mu_{T_{2,3,4}} = e^{\left(P_{S_{2,3,4}} + \frac{1}{2}Q_{S_{2,3,4}}{}^2\right)} = e^{(23.965 + (0.5)(1.027))}$, which is $E[T_{2,3,4}] = \mu_{T_{2,3,4}} = 42744227758$.

Now we can calculate $\rho_{Y_3{Y_2}^{1/c},Y_4{Y_2}^{1/c}} = \frac{E\left[Y_3Y_4{Y_2}^{\frac{2}{c}}\right] - \mu_{V_3}\mu_{V_4}}{\sigma_{V_3}\sigma_{V_4}}$, which is

$\rho_{Y_3{Y_2}^{1/c},Y_4{Y_2}^{1/c}} = \frac{42744227758 - (347348.652)(100760.716)}{(1.953E+05)(5.647E+04)} = 0.5989$.

This process must be repeated for all i, j to compute $\boldsymbol{\rho_V}$ as:

$$\boldsymbol{\rho_Y} = \begin{bmatrix} 1 & 0.7036 & 0.7264 & 0.8180 & 0.8609 & 0.7682 & 0.8543 & 0.7877 & 0.7351 \\ 0.7036 & 1 & 0.5989 & 0.6578 & 0.6761 & 0.6292 & 0.6779 & 0.6394 & 0.6039 \\ 0.7264 & 0.5989 & 1 & 0.6704 & 0.6909 & 0.6401 & 0.6919 & 0.6510 & 0.6141 \\ 0.8180 & 0.6578 & 0.6704 & 1 & 0.7702 & 0.7067 & 0.7695 & 0.7203 & 0.6776 \\ 0.8609 & 0.6761 & 0.6909 & 0.7702 & 1 & 0.7297 & 0.7992 & 0.7450 & 0.6993 \\ 0.7682 & 0.6292 & 0.6401 & 0.7067 & 0.7297 & 1 & 0.7302 & 0.6858 & 0.6462 \\ 0.8543 & 0.6779 & 0.6919 & 0.7695 & 0.7992 & 0.7302 & 1 & 0.7450 & 0.6999 \\ 0.7877 & 0.6394 & 0.6510 & 0.7203 & 0.7450 & 0.6858 & 0.7450 & 1 & 0.6577 \\ 0.7351 & 0.6039 & 0.6141 & 0.6776 & 0.6993 & 0.6462 & 0.6999 & 0.6577 & 1 \end{bmatrix}$$

Performing the calculation, $\sigma_V = \sqrt{\boldsymbol{\sigma_V}^{\boldsymbol{T}} \boldsymbol{\rho_V} \boldsymbol{\sigma_V}} = 1137353.64$.

Since we know μ_V and σ_V, we can calculate the parameters of the underlying normal distribution P_U and Q_U so we can calculate $\mathrm{E}[Y_1 Y_2]$.

$$\mathrm{bE}[Y_1 Y_2] = bE\left[\left(\sum_{i=2}^{10} Y_i Y_2^{\frac{1}{c}}\right)^c\right] = bE[V^c] = be^{\left(cP_U + \frac{1}{2}Q_U{}^2 c^2\right)} =$$
$$(0.498)e^{\left((0.9)(14.46) + \frac{1}{2}((0.9)(0.498))^2\right)} = 245930$$

8.2.1.4 Computing the Type I-2 Functional Correlation

In step 3, we find the functional correlation ρ_{Y_1,Y_2} by combining terms into the expression found in Step 1.

$$\rho_{Y_1,Y_2} = \frac{\mathrm{E}[Y_1 Y_2] - \mathrm{E}[Y_1]\mathrm{E}[Y_2]}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}} = \frac{245930 - (413.17)(572.71)}{(201.05)(177.02)} = 0.2614$$

8.3 Type II-1 Functional Correlation

In this case, we have two CERs Y_1 and Y_2 expressed as functions of the same random variable, X.

$Y_\mathrm{i} = f_\mathrm{i}(X)\varepsilon_\mathrm{i} = (a_\mathrm{i} + b_\mathrm{i} X^{c_\mathrm{i}})\varepsilon_\mathrm{i}$; where **8-2**
a_i, b_i, and c_i are coefficients of the CERs with ($Var(\cdot) = 0$),
ε_i are multiplicative errors of the CERs with $\mu_{\varepsilon_\mathrm{i}} = 1$, and a given value of $\sigma_{\varepsilon_\mathrm{i}}$
$\rho_{f_\mathrm{i},\varepsilon_\mathrm{i}} = 0$, since CERs and their errors are assumed to be independent.

We can find the Type II-1 functional correlation between these CERs since they share a common variable, X. The correlation between these two CERs is ρ_{Y_1,Y_2}, and based from Step 1.

$$\rho_{Y_1,Y_2} = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}} \quad \textbf{8-3}$$

Clearly, we will need to find the formulae for $\mathrm{Cov}(Y_1, Y_2)$ and $Var(Y_\mathrm{i})$ to find ρ_{Y_1,Y_2}. Using Equation 7-6 from the propagation of errors method, the standard deviation of Y_i is $\sigma_{Y_i} = \sqrt{Var(Y_i)}$, where $(Y_\mathrm{i}) = (\mu_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}})^2 + (\mu_{\varepsilon_\mathrm{i}}\sigma_{f_\mathrm{i}})^2 + (\sigma_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}})^2$.

If Y_1 and Y_2 are CERs with multiplicative errors, then $\mu_{\varepsilon_\mathrm{i}} = 1$ and we know $\sigma_{\varepsilon_\mathrm{i}}$ from the percent standard error of the CER. $Var(Y_\mathrm{i})$ reduces to:

$$Var(Y_\mathrm{i}) = (\mu_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}})^2 + (\sigma_{f_\mathrm{i}})^2 + (\sigma_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}})^2 \qquad \textbf{8-4}$$

The terms $\mu_{f_\mathrm{i}}, \sigma_{f_\mathrm{i}}$ are computed from Equations 4-28 and 4-29 as follows:

$$\mu_{f_\mathrm{i}} = a_\mathrm{i} + b_\mathrm{i}E[X^{c_\mathrm{i}}] \qquad \textbf{8-5}$$

$$\sigma_{f_\mathrm{i}} = \sqrt{b_\mathrm{i}^2 Var(X^{c_\mathrm{i}})} = b_\mathrm{i}\sqrt{Var(X^{c_\mathrm{i}})} \qquad \textbf{8-6}$$

$$Var(Y_\mathrm{i}) = \sigma_{\varepsilon_\mathrm{i}}^2\left(a_\mathrm{i}^2 + 2b_\mathrm{i}E[X^{c_\mathrm{i}}] + b_\mathrm{i}^2E[X^{c_\mathrm{i}}]^2 + b_\mathrm{i}^2Var(X^{c_\mathrm{i}})\right) + b_\mathrm{i}^2Var(X^{c_\mathrm{i}}) \qquad \textbf{8-7}$$

Using the results from Section 4.3.3 and assuming X is a triangular distribution, $X = T(L, M, H)$, then:

$$\sigma_{f_1} = b_\mathrm{i}\sqrt{\frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{2c_\mathrm{i}+2}-L^{2c_\mathrm{i}+2}}{2c_\mathrm{i}+2} - L\frac{M^{2c_\mathrm{i}+1}-L^{2c_\mathrm{i}+1}}{2c_\mathrm{i}+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{2c_\mathrm{i}+1}-M^{2c_\mathrm{i}+1}}{2c_\mathrm{i}+1} - \frac{H^{2c_\mathrm{i}+2}-M^{2c_\mathrm{i}+2}}{2c_\mathrm{i}+2}\right\}\right] - \left(\frac{\mu_{f_\mathrm{i}}-a_\mathrm{i}}{b_\mathrm{i}}\right)^2}$$

We need to calculate μ_{Y_i} and σ_{Y_i}.

$$\mu_{Y_\mathrm{i}} = E[Y_i] = E[f_\mathrm{i}\varepsilon_\mathrm{i}] = \mu_{f_\mathrm{i}}\mu_{\varepsilon_\mathrm{i}} + \rho_{f_\mathrm{i},\varepsilon_\mathrm{i}}\sigma_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}}$$

Since $\mu_{\varepsilon_\mathrm{i}} = 1$ and $\rho_{f_\mathrm{i},\varepsilon_\mathrm{i}}\sigma_{f_\mathrm{i}}\sigma_{\varepsilon_\mathrm{i}} = 0$, then $\mu_{Y_\mathrm{i}} = \mu_{f_\mathrm{i}}$.

The standard deviation of Y_i is calculated using the propagation of errors method:

$$\sigma_{Y_\mathrm{i}} = \sqrt{\sigma_{f_\mathrm{i}}^2 + \sigma_{\varepsilon_\mathrm{i}}^2\mu_{f_\mathrm{i}}^2 + \sigma_{f_\mathrm{i}}^2\sigma_{\varepsilon_\mathrm{i}}^2}\text{, and } \mu_{Y_\mathrm{i}} = \mu_{f_\mathrm{i}}\mu_{\varepsilon_\mathrm{i}}$$

The Type II-1 correlation between the CERs is:

$$\rho_{Y_1,Y_2} = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}} = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sqrt{\sigma_{Y_1}^2}\sqrt{\sigma_{Y_2}^2}} = \frac{\mathrm{E}[Y_1Y_2]-\mathrm{E}[Y_1]\mathrm{E}[Y_2]}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{\mathrm{E}[Y_1Y_2]-\mu_{Y_1}\mu_{Y_2}}{\sigma_{Y_1}\sigma_{Y_2}}$$

$$\rho_{Y_1,Y_2} = \frac{\mathrm{E}[Y_1Y_2]-\mu_{Y_1}\mu_{Y_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_\mathrm{i}}^2+\sigma_{\varepsilon_\mathrm{i}}^2\mu_{f_\mathrm{i}}^2+\sigma_{f_\mathrm{i}}^2\sigma_{\varepsilon_\mathrm{i}}^2}\right)}$$

Using Step 2c,

$Y_1Y_2 = [(a_1 + b_1x^{c_1})\varepsilon_1][(a_2 + b_2x^{c_2})\varepsilon_2] = (\varepsilon_1a_1 + \varepsilon_1b_1x^{c_1})(\varepsilon_2a_2 + \varepsilon_2b_2x^{c_2})$

Multiplication of terms produces:

$Y_1Y_2 = a_1a_2\varepsilon_1\varepsilon_2 + a_1b_2\varepsilon_1\varepsilon_2x^{c_2} + a_2b_1\varepsilon_1\varepsilon_2x^{c_1} + \varepsilon_1\varepsilon_2b_1b_2x^{c_1}x^{c_2}$

Calculating the expectation of the terms in Step 2d:

$\mathrm{E}[Y_1Y_2] = E[a_1a_2\varepsilon_1\varepsilon_2] + E[a_1b_2\varepsilon_1\varepsilon_2x^{c_2}] + E[a_2b_1\varepsilon_1\varepsilon_2x^{c_1}] + E[\varepsilon_1\varepsilon_2b_1b_2x^{c_1}x^{c_2}]$

Separating constant scaling terms:

$\mathrm{E}[Y_1Y_2] = a_1a_2E[\varepsilon_1\varepsilon_2] + a_1b_2E[\varepsilon_1\varepsilon_2x^{c_2}] + a_2b_1E[\varepsilon_1\varepsilon_2x^{c_1}] + b_1b_2E[\varepsilon_1\varepsilon_2x^{c_1}x^{c_2}]$

Expectations with the product $[\varepsilon_1\varepsilon_2]$ appear consistently, so we will define the product as ω, such that

$E[\varepsilon_1\varepsilon_2] = E[\omega] = \mu_{\varepsilon_1}\mu_{\varepsilon_2} + \rho_{\varepsilon_1\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} = 1 + \rho_{\varepsilon_1\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$, which is a constant defined by the CER.

So, $\mathrm{E}[Y_1Y_2] = a_1a_2E[\omega] + a_1b_2E[\omega x^{c_2}] + a_2b_1E[\omega x^{c_1}] + b_1b_2E[\omega x^{c_1+c_2}]$.

We need to find $E[\omega x^{\mathrm{k}}] = \rho_{\omega,x^{\mathrm{k}}}\sigma_\omega\sigma_{x^{\mathrm{k}}} + E[\omega]\, E[x^{\mathrm{k}}]$.

Assume $\rho_{\omega,x^{\mathrm{k}}}\sigma_\omega\sigma_{x^{\mathrm{k}}} = 0$, so $E[\omega x^{\mathrm{k}}] = \left(1 + \rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right) E[x^{\mathrm{k}}]$ and

$\mathrm{E}[Y_1Y_2] = a_1a_2E[\omega] + a_1b_2E[\omega]\, E[x^{c_2}] + a_2b_1\, E[\omega]\, E[x^{c_1}] + b_1b_2E[\omega]\, E[x^{c_1+c_2}]$

$\mathrm{E}[Y_1Y_2] = E[\omega](a_1a_2 + a_1b_2\, E[x^{c_2}] + a_2b_1\, E[x^{c_1}] + b_1b_2\, E[x^{c_1+c_2}])$

$\mathrm{E}[Y_1Y_2] = \left(1 + \rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)(a_1a_2 + a_1b_2\, E[x^{c_2}] + a_2b_1\, E[x^{c_1}] + b_1b_2\, E[x^{c_1+c_2}])$

And we know $E[\omega] = \left(1 + \rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)$ and

$$E[x^{\mathrm{k}}] = \frac{2}{(H-L)(M-L)}\left\{\frac{M^{k+2}-L^{k+2}}{k+2} - L\,\frac{M^{k+1}-L^{k+1}}{k+1}\right\} + \frac{2}{(H-L)(H-M)}\left\{H\,\frac{H^{k+1}-M^{k+1}}{k+1} - \frac{H^{k+2}-M^{k+2}}{k+2}\right\}$$

We can solve $E[x^{c_1}]$, $E[x^{c_2}]$, and $E[x^{c_1+c_2}]$ using formulas for $E[x^{\mathrm{k}}]$ and substituting k for $c_1, c_2, c_1{+}c_2$. Formulas for $E[X^{\mathrm{k}}]$ for different distribution types are located in Appendix A – Probability Distributions.

So if X is defined by a triangular distribution, then we have the following for Step 3:

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})(a_1a_2+a_1b_2\,E[x^{c_2}]+a_2b_1\,E[x^{c_1}]+b_1b_2\,E[x^{c_1+c_2}])-\mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$

Of course, not every function will have the form, $Y_i = f_i(x)\varepsilon_i = (a_i + b_i x^{c_i})\varepsilon_i$, so we will consider three simplified cases.

Case 1: if $c_1 = 1$, and $c_2 = 1$ then $Y_i = (a_i + b_i x)\varepsilon_i$

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})(a_1a_2+\mu_x[a_1b_2+a_2b_1]+b_1b_2\,E[x^2])-\mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$

Case 2: if $a_1 = 0$, and $a_2 = 0$ then $Y_i = b_i x^{c_i}\varepsilon_i$

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})(b_1b_2\,E[x^{c_1+c_2}])-\mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$

Special Case 3: if $a_1 = 0$, and $a_2 = 0$; and $\sigma_{\varepsilon_1} = 0$ and $\sigma_{\varepsilon_2} = 0$; then $Y_i = b_i x^{c_i}$, which is the case from Garvey (2000).[51]

$$\rho_{Y_1,Y_2} = \frac{(b_1b_2\,E[x^{c_1+c_2}])-\mu_{f_1}\mu_{f_2}}{\sigma_{f_1}\sigma_{f_2}}$$

8.3.1 Common Predecessor Functional Correlation

In the case of a schedule network with parallel tasks, we are faced with the situation whereby we must compute the functional correlation between two tasks T1 and T2 that have the same predecessor, P, that has a finish date F_P. Assume the durations of T1 and T2 (D_1 and D_2, respectively) are correlated by ρ_{D_1,D_2}. The start dates of T1 and T2 are F_1 and F_2 respectively. The finish dates of T1 and T2 are $F_1 = F_P + D_1$ and $F_2 = F_P + D_2$. The resulting standard deviations of the finish dates are $\sigma_{F_1} = \sqrt{F_P{}^2 + D_1{}^2}$ and $\sigma_{F_2} = \sqrt{F_P{}^2 + D_2{}^2}$.

Using Step 1, the correlation between F_1 and F_2 is expressed mathematically as:

$$\rho_{F_1,F_2} = \frac{E[F_1F_2]-E[F_1]E[F_2]}{\sigma_{F_1}\sigma_{F_2}}$$

Step 2a: if $\mu_{F_1} = \mu_{F_P} + \mu_{D_1}$ and $\mu_{F_2} = \mu_{F_P} + \mu_{D_2}$, then

$$\mu_{F_1}\mu_{F_2} = \mu_{F_P}{}^2 + \mu_{F_P}\mu_{D_2} + \mu_{F_P}\mu_{D_1} + \mu_{D_1}\mu_{D_2} + \rho_{D_1,D_2}\sigma_{D_1}\sigma_{D_2}.$$

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[51] Garvey, P. R. (2000). Probability Methods for Cost Uncertainty Analysis: A Systems Engineering Perspective. New York, NY: Marcel Dekker.

Step 2b: the standard deviations of F_1 and F_2 are

$$\sigma_{F_1} = \sqrt{{\sigma_{F_P}}^2 + {\sigma_{D_1}}^2} \text{ and } \sigma_{F_2} = \sqrt{{\sigma_{F_P}}^2 + {\sigma_{D_2}}^2}.$$

Step 2c: The first expectation term requires expansion of the product F_1F_2, which is

$$F_1F_2 = (F_P + D_1)(F_P + D_2) = {F_P}^2 + F_PD_2 + F_PD_1 + D_1D_2$$

Step 2d: then the product moment is

$$E[F_1F_2] = E\left[{F_P}^2\right] + E[F_P]E[D_2] + E[F_P]E[D_1] + E[D_1D_2]$$

Since $E\left[{F_P}^2\right] = {\mu_{F_P}}^2 + {\sigma_{F_P}}^2$,

$$E[F_1F_2] = {\mu_{F_P}}^2 + {\sigma_{F_P}}^2 + \mu_{F_P}\mu_{D_2} + \mu_{F_P}\mu_{D_1} + \mu_{D_1}\mu_{D_2} + \rho_{D_1,D_2}\sigma_{D_1}\sigma_{D_2}$$

Step 3: the correlation between the two finish dates is then

$$\rho_{F_1,F_2} = \frac{{\mu_{F_P}}^2 + {\sigma_{F_P}}^2 + \mu_{F_P}\mu_{D_2} + \mu_{F_P}\mu_{D_1} + \mu_{D_1}\mu_{D_2} + \rho_{D_1,D_2}\sigma_{D_1}\sigma_{D_2} - {\mu_{F_P}}^2 - \mu_{F_P}\mu_{D_2} - \mu_{F_P}\mu_{D_1} - \mu_{D_1}\mu_{D_2}}{\sigma_{F_1}\sigma_{F_2}}$$

Through cancellation of terms, we arrive at Equation 8-8 - a useful relationship in schedule uncertainty analysis.

$$\rho_{F_1,F_2} = \frac{{\sigma_{F_P}}^2 + \rho_{D_1,D_2}\sigma_{D_1}\sigma_{D_2}}{\sqrt{{\sigma_{F_P}}^2 + {\sigma_{D_1}}^2}\sqrt{{\sigma_{F_P}}^2 + {\sigma_{D_2}}^2}}, \qquad \textbf{8-8}$$

8.3.2 Type II-1 Functional Correlation Example

For this example we will calculate the functional correlation between two CERs (Y_2 and Y_3) that share a common cost driver ($X = X_{2b} = X_3$), which is defined as the frequency of operation. The CERs are defined as:

$$Y_2 = 34.36{X_{2a}}^{0.5}{X_{2b}}^{0.8}\varepsilon_2 \text{ and } Y_3 = 30.06{X_3}^{0.8}\varepsilon_3.$$

They share the random variable, X, where $X = T(16,17,18)$; and the CER uncertainties are $\sigma_{\varepsilon_2} = 0.3, \sigma_{\varepsilon_3} = 0.4$, and $\rho_{\varepsilon_1,\varepsilon_2} = 0.2$. The other driver of CER Y_2 is X_{2a}, which is defined by a triangular distribution $T(2,3,4)$.

When statistically summing these CERs in a WBS we need to find the functional correlation, ρ_{Y_2,Y_3}.

In the first step of the calculation process, we define the correlation between the CERs as

$\rho_{Y_2,Y_3} = \frac{E[Y_2Y_3]-E[Y_2]E[Y_3]}{\sigma_{Y_2}\sigma_{Y_3}}$

In Step 2a, we find the means of $Y_2$and Y_3.

$E[Y_2] = 34.36E\left[{X_{2a}}^{0.5}\right]E\left[{X_{2b}}^{0.8}\right]E[\varepsilon_2]$, and $[Y_3] = 30.06E\left[{X_3}^{0.8}\right]E[\varepsilon_3]$

Since $E[\varepsilon_2] = 1$ and $E[\varepsilon_3] = 1$,

$E[Y_2] = 34.36E\left[{X_{2a}}^{0.5}\right]E\left[{X_{2b}}^{0.8}\right]$, and $E[Y_3] = 30.06E\left[{X_3}^{0.8}\right]$

Using the relationship for the expectation of a triangular PDF raised to a power, k, and substituting the parameters of the triangular PDF, we get

$E\left[{X_{2a}}^{0.5}\right] = 1.728$, $E\left[{X_{2b}}^{0.8}\right] = 9.646$, and through similarity $E\left[{X_3}^{0.8}\right] = 9.646$

The means of Y_2 and Y_3 are, therefore,

$E[Y_2] = (34.36)(1.728)(9.646) = 572.706$, and

$E[Y_3] = (30.06)(9.646) = 289.953$.

In Step 2b, we find the standard deviations of Y_2 and Y_3. Using the relationship for the variance of a triangular PDF raised to a power, k, and substituting the parameters of the triangular PDF, we get

$Var\left({X_{2a}}^{0.5}\right) = 0.01407$, $Var\left({X_{2b}}^{0.8}\right) = 0.03435$, and $Var\left({X_3}^{0.8}\right) = 0.03435$.

We need to combine the independent variables in CER Y_2 to find $Var\left(f_{X_2}\right)$.

$Var\left(f_{X_2}\right) =$
$(34.36)^2\left[E^2\left[{X_{2b}}^{0.8}\right]Var\left({X_{2a}}^{0.5}\right) + E^2\left[{X_{2a}}^{0.5}\right]Var\left({X_{2b}}^{0.8}\right) + Var\left({X_{2a}}^{0.5}\right)Var\left({X_{2b}}^{0.8}\right)\right]$

This results in $Var\left(f_{X_2}\right) = 1667.360$. Combining $Var\left(f_{X_2}\right)$ with the variance of the error term using the propagation of errors method results in:

$Var(Y_2) = \left[Var\left(f_{X_2}\right) + E^2\left(f_{X_2}\right)Var(\varepsilon_2) + Var\left(f_{X_2}\right)Var(\varepsilon_2)\right] = 31336.746$

Similarly,

$Var(Y_3) = \left[Var\left(f_{X_3}\right) + E^2\left(f_{X_3}\right)Var(\varepsilon_3) + Var\left(f_{X_3}\right)Var(\varepsilon_3)\right] = 13487.670$.

$\sigma_{Y_2} = \sqrt{31336.746} = 177.0219$ and $\sigma_{Y_3} = \sqrt{13487.670} = 116.136$

In Step 2c, we find the product Y_2Y_3, which is

$$Y_2Y_3 = \left(34.36{X_{2a}}^{0.5}{X_{2b}}^{0.8}\varepsilon_2\right)\left(30.06{X_3}^{0.8}\varepsilon_3\right) = (34.36)(30.06)\left({X_{2a}}^{0.5}\right)\left({X_{2b}}^{1.6}\right)(\varepsilon_2\varepsilon_3)$$

$$Y_2Y_3 = 1032.862\left({X_{2a}}^{0.5}\right)\left({X_{2b}}^{1.6}\right)(\varepsilon_2\varepsilon_3)$$

$$E\left[{X_{2b}}^{1.6}\right] = 93.076$$

Following Step 2d, the expectation of this product is

$E[Y_2Y_3] = 1032.862E\left[{X_{2a}}^{0.5}\right]E\left[{X_{2b}}^{1.6}\right]E[\varepsilon_2\varepsilon_3]$, and $E[\varepsilon_2\varepsilon_3] = 1 + \rho_{\varepsilon_2,\varepsilon_3}\sigma_{\varepsilon_2}\sigma_{\varepsilon_3}$

Using inputs and previously calculated values, this becomes

$E[Y_2Y_3] = (1032.862)(1.728)(93.076)\left(1 + (0.2)(0.3)(0.4)\right) = 170106.250$

The product $E[Y_2]E[Y_3]$ is

$$E[Y_2]E[Y_3] = (572.706)(289.953) = 166058.082$$

Combining these values into ρ_{Y_1,Y_2} results in

$$\rho_{Y_1,Y_2} = \frac{E[Y_2Y_3] - E[Y_2]E[Y_3]}{\sigma_{Y_2}\sigma_{Y_3}} = \frac{170106.250 - 166058.082}{(177.022)(116.136)} = \frac{4048.168}{19959.751} = 0.1969$$

8.3.3 Type II-1 Functional Correlation between Multivariate Functions

What is the correlation between two CERs that have two RVs and share one RV in common?

$Y_1 = f_1(\mathrm{v},\mathrm{w})\varepsilon_\mathrm{i} = (a_1 + b_1x^{c_1}w^{d_1})\varepsilon_1$, and $Y_2 = f_2(\mathrm{u},\mathrm{w})\varepsilon_\mathrm{i} = (a_2 + b_2x^{c_2}u^{d_2})\varepsilon_2$; where

a_1, b_1, and c_1 are coefficients of the CERs with $(Var(\cdot) = 0)$,

ε_i are multiplicative errors of the CERs with $\mu_{\varepsilon_\mathrm{i}} = 1$, and

$\rho_{f_\mathrm{i},\varepsilon_\mathrm{i}} = 0$, since CERs and their errors are assumed to be independent.

$$\rho_{Y_1,Y_2} = \frac{\mathrm{E}[Y_1Y_2] - \mu_{Y_1}\mu_{Y_2}}{\sigma_{Y_1}\sigma_{Y_2}}$$

$Y_1Y_2 = [(a_1 + b_1x^{c_1}w^{d_1})\varepsilon_1][(a_2 + b_2x^{c_2}u^{d_2})\varepsilon_2] = (\varepsilon_1a_1 + \varepsilon_1b_1x^{c_1}w^{d_1})(\varepsilon_2a_2 + \varepsilon_2b_2x^{c_2}u^{d_2})$

$Y_1Y_2 = \varepsilon_1\varepsilon_2a_1a_2 + \varepsilon_1\varepsilon_2a_1b_2x^{c_2}u^{d_2} + \varepsilon_1\varepsilon_2a_2b_1x^{c_1}w^{d_1} + \varepsilon_1\varepsilon_2b_1b_2x^{c_1}w^{d_1}x^{c_2}u^{d_2}$

$E[Y_1Y_2] = a_1a_2E[\varepsilon_1\varepsilon_1] + E[\varepsilon_1\varepsilon_1]a_1b_2E[x^{c_2}u^{d_2}] + E[\varepsilon_1\varepsilon_1]a_2b_1E[x^{c_1}w^{d_1}] + E[\varepsilon_1\varepsilon_1]b_1b_2E[x^{c_1}w^{d_1}x^{c_2}u^{d_2}]$

$E[\varepsilon_1\varepsilon_2] = E[\omega] = \mu_{\varepsilon_1}\mu_{\varepsilon_2} + \rho_{\varepsilon_1\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2} = 1 + \rho_{\varepsilon_1\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$, and

$E[Y_1Y_2] =$
$E[\omega]\{a_1a_2 + a_1b_2E[x^{c_2}]E[u^{d_2}] + a_2b_1E[x^{c_1}]E[w^{d_1}] + b_1b_2E[x^{c_1+c_2}]E[w^{d_1}]E[u^{d_2}]\}$

$\sigma_{Y_i} = \sqrt{\sigma_{f_i}{}^2 + \sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2 + \sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}$, and $\mu_{Y_i} = \mu_{f_i}$

$\mu_{f_1} = a_1 + b_1E[x^{c_1}w^{d_1}]$, and $\mu_{f_2} = a_2 + b_2E[x^{c_2}u^{d_2}]$

$\sigma_{f_1} = b_1\sqrt{Var(x^{c_1}w^{d_1})}$, and $\sigma_{f_2} = b_2\sqrt{Var(x^{c_2}u^{d_2})}$

$Var(w^{c_1}x^{d_1}) = E[x^{2c_1}w^{2d_1}] - (E[x^{c_1}w^{d_1}])^2 = E[x^{2c_1}]E[w^{2d_1}] - (E[x^{c_1}]E[w^{d_1}])^2$

$\sigma_{f_1} = b_1\sqrt{E[x^{2c_1}]E[w^{2d_1}] - (E[x^{c_1}]E[w^{d_1}])^2}$

$Var(u^{c_2}w^{d_2}) = E[x^{2c_2}u^{2d_2}] - (E[x^{c_2}u^{d_2}])^2 = E[x^{2c_2}]E[u^{2d_2}] - (E[x^{c_2}]E[u^{d_2}])^2$

$$\sigma_{f_2} = b_2\sqrt{E[x^{2c_2}]E[u^{2d_2}] - (E[x^{c_2}]E[u^{d_2}])^2}$$

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})\{a_1a_2+a_1b_2E[x^{c_2}]E[u^{d_2}]+a_2b_1E[x^{c_1}]E[w^{d_1}]+b_1b_2E[x^{c_1+c_2}]E[w^{d_1}]E[u^{d_2}]\} - \mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$

If u and w are constants; and if $u = 1$ and $w = 1$, then

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})\{a_1a_2+a_1b_2E[x^{c_2}](1)+a_2b_1E[x^{c_1}](1)+b_1b_2E[x^{c_1+c_2}](1)(1)\} - \mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$

$$\rho_{Y_1,Y_2} = \frac{(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2})\{a_1a_2+a_1b_2E[x^{c_2}]+a_2b_1E[x^{c_1}]+b_1b_2E[x^{c_1+c_2}]\} - \mu_{f_1}\mu_{f_2}}{\prod_{i=1}^{2}\left(\sqrt{\sigma_{f_i}{}^2+\sigma_{\varepsilon_i}{}^2\mu_{f_i}{}^2+\sigma_{f_i}{}^2\sigma_{\varepsilon_i}{}^2}\right)}$$, which is the same result as for the single variable CER cases.

8.4 Type II-2 Functional Correlation

This type of functional correlation occurs when two nested functions share one or more RVs in common. This occurs in a resource-loaded schedule where costs are derived from particular task durations.

Consider a simple case of the cost of a project with three WBS elements where the total cost is the value X_{Tot}.

$X_{Tot} = X_1 + X_2 + X_4$, where X_i is the cost of WBS element i.

Now consider the schedule duration of the project, D, where its total duration is

$D_{Tot} = D_1 + D_3 + D_4$, where D_i is the cost of task i.

We also know that the costs of WBS elements 1, 2, and 4 are their respective durations multiplied by a rate, r_i, where $X_i = D_i r_i$.

Following Step 1 of the functional correlation calculation process, the correlation between total cost and total schedule duration can be expressed as:

$$\rho_{X_{Tot},D_{Tot}} = \frac{E[X_{Tot}D_{Tot}] - E[X_{Tot}]E[D_{Tot}]}{\sigma_{X_{Tot}}\sigma_{D_{Tot}}}$$

In Steps 2a and 2b we calculate $E[X_{Tot}]$, $[D_{Tot}]$, $\sigma_{X_{Tot}}$, and $\sigma_{D_{Tot}}$.

In Step 2c the product $X_{Tot}D_{Tot}$ is

$$X_{Tot}D_{Tot} = X_1(D_1 + D_3 + D_4) + X_2(D_1 + D_3 + D_4) + X_4(D_1 + D_3 + D_4)$$

$$X_{Tot}D_{Tot} = X_1D_1 + X_1D_3 + X_1D_4 + X_2D_1 + X_2D_3 + X_2D_4 + X_4D_1 + X_4D_3 + X_4D_4$$

In Step 2d, we calculate

$E[X_{Tot}D_{Tot}] = \mu_{X_1}\mu_{D_1} + \mu_{X_1}\mu_{D_3} + \mu_{X_1}\mu_{D_4} + \mu_{X_2}\mu_{D_1} + \mu_{X_2}\mu_{D_3} + \mu_{X_2}\mu_{D_4} + \mu_{X_4}\mu_{D_1} + \mu_{X_4}\mu_{D_3} + \mu_{X_4}\mu_{D_4}$

and

$E[X_{Tot}]E[D_{Tot}] = E[X_1D_1] + E[X_1D_3] + E[X_1D_4] + E[X_2D_1] + E[X_2D_3] + E[X_2D_4] + E[X_4D_1] + E[X_4D_3] + E[X_4D_4]$

For each pair X_i and D_j, the term $E\left[X_iD_j\right] = \mu_{X_i}\mu_{D_j} + \rho_{X_i,D_j}\sigma_{X_i}\sigma_{D_j}$

By inspection we see the only remaining terms in $E[X_{Tot}D_{Tot}] - E[X_{Tot}]E[D_{Tot}]$ will be the sum of all pairs of $\rho_{X_i,D_j}\sigma_{X_i}\sigma_{D_j}$. Let us assume for simplicity that $\rho_{X_i,D_j} = 1$ for $i = j$ and $\rho_{X_i,D_j} = 0$ for $i \neq j$. This reduces the numerator of the correlation expression in Step 1 to

$$E[X_{Tot}D_{Tot}] - E[X_{Tot}]E[D_{Tot}] = \rho_{X_1,D_1}\sigma_{X_1}\sigma_{D_1} + \rho_{X_4,D_4}\sigma_{X_4}\sigma_{D_4}$$

Dividing by the product $\sigma_{X_{Tot}}\sigma_{D_{Tot}}$ we have

$$\rho_{X_{Tot},D_{Tot}} = \frac{\rho_{X_1,D_1}\sigma_{X_1}\sigma_{D_1} + \rho_{X_4,D_4}\sigma_{X_4}\sigma_{D_4}}{\sigma_{X_{Tot}}\sigma_{D_{Tot}}}$$

Since $X_i = D_i r_i$, we can reduce this correlation to a combination of rates and task durations

$$\rho_{X_{Tot},D_{Tot}} = \frac{\rho_{X_1,D_1} r_1 {\sigma_{D_1}}^2 + \rho_{X_4,D_4} r_4 {\sigma_{D_4}}^2}{\sigma_{X_{Tot}} \sigma_{D_{Tot}}}$$

We see from this example that if schedule durations in the critical path are uncorrelated, they drop from the numerator of the expression of total cost and schedule correlation and it becomes a sum of covariance terms.

8.5 Type III-1 Functional Correlation

Type III functional correlation exists between pairs of random variables such as two CERs Y_1 and Y_2 that share a partially-dependent random variable such as their multiplicative errors. In this case we wish to find

ρ_{Y_1,Y_2} , where $Y_1 = (a_1 + b_1 {X_1}^{c_1})\varepsilon_1$, $Y_2 = (a_2 + b_2 {X_2}^{c_2})\varepsilon_2$, and $\rho_{\varepsilon_1,\varepsilon_2} \neq 0$

The formula used to determine the correlation coefficient from Step 1is

$$\rho_{X,Y} = \frac{\mathrm{E}[Y_1 Y_2] - \mathrm{E}[Y_1]\mathrm{E}[Y_2]}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}}$$
$$= \frac{\mathrm{E}[(a_1 + b_1 {X_1}^{c_1})\varepsilon_1 (a_2 + b_2 {X_2}^{c_2})\varepsilon_2] - \mathrm{E}[(a_1 + b_1 {X_1}^{c_1})\varepsilon_1]\mathrm{E}[(a_2 + b_2 {X_2}^{c_2})\varepsilon_2]}{\sqrt{Var((a_1 + b_1 {X_1}^{c_1})\varepsilon_1)}\sqrt{Var((a_2 + b_2 {X_2}^{c_2})\varepsilon_2)}}$$
$$= \frac{\mathrm{E}[(a_1 + b_1 {X_1}^{c_1})\varepsilon_1 (a_2 + b_2 {X_2}^{c_2})\varepsilon_2] - \mathrm{E}[(a_1 + b_1 {X_1}^{c_1})\varepsilon_1]\mathrm{E}[(a_2 + b_2 {X_2}^{c_2})\varepsilon_2]}{b_1 b_2 \sqrt{Var(({X_1}^{c_1})\varepsilon_1)}\sqrt{Var(({X_2}^{c_2})\varepsilon_2)}}$$

Using Step 2a, from Equation 8-5, $\mathrm{E}[Y_i] = a_\mathrm{i} + b_\mathrm{i} E[X^{c_\mathrm{i}}]$

Step 2b, from Equation 8-6 shows, $\sigma_{Y_i} = b_\mathrm{i}\sqrt{Var({X_i}^{c_\mathrm{i}}\varepsilon_i)}$. Since ${X_i}^{c_\mathrm{i}}$ and ε_i are uncorrelated, we use the propagation of errors method, which results in:

$$\sigma_{Y_i} = b_\mathrm{i}\sqrt{[E^2({X_i}^{c_\mathrm{i}})Var(\varepsilon_i)] + [Var({X_i}^{c_\mathrm{i}})] + [Var({X_i}^{c_\mathrm{i}})Var(\varepsilon_i)]}$$

Expanding the product of the variables ($Y_1 Y_2$) in Step 2c results in:

$$Y_1 Y_2 = a_1 a_2 \varepsilon_1 \varepsilon_2 + a_1 b_2 \varepsilon_1 \varepsilon_2 {X_2}^{c_2} + a_2 b_1 \varepsilon_1 \varepsilon_2 {X_1}^{c_1} + \varepsilon_1 \varepsilon_2 b_1 b_2 {X_1}^{c_1} {X_2}^{c_2}$$

Taking the expectation of the product in Step 2d,

$$\mathrm{E}[Y_1 Y_2] = a_1 a_2 E[\varepsilon_1 \varepsilon_2] + a_1 b_2 E[\varepsilon_1 \varepsilon_2 {X_2}^{c_2}] + a_2 b_1 E[\varepsilon_1 \varepsilon_2 {X_1}^{c_1}] + b_1 b_2 E[\varepsilon_1 \varepsilon_2 {X_1}^{c_1} {X_2}^{c_2}]$$

$$\mathrm{E}[Y_1 Y_2] = a_1 a_2 E[\varepsilon_1 \varepsilon_2] + a_1 b_2 E[\varepsilon_1 \varepsilon_2] E[{X_2}^{c_2}] + a_2 b_1 E[\varepsilon_1 \varepsilon_2] E[{X_1}^{c_1}] + b_1 b_2 E[\varepsilon_1 \varepsilon_2] E[{X_1}^{c_1}] E[{X_2}^{c_2}]$$

Since $E[\varepsilon_1 \varepsilon_2] = \mu_{\varepsilon_1}\mu_{\varepsilon_1} + \rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$, we can reduce this to $E[\varepsilon_1 \varepsilon_2] = 1 + \rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}$.

This results in the expectation term

$$\mathrm{E}[Y_1Y_2] = a_1a_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right) + a_1b_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_2}^{c_2}] + a_2b_1\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}] + b_1b_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}]E[{X_2}^{c_2}]$$

$$\mathrm{E}[Y_1]\mathrm{E}[Y_2] = (a_1 + b_1E[{X_1}^{c_1}])(a_2 + b_2E[{X_2}^{c_2}])$$

$$\mathrm{E}[Y_1]\mathrm{E}[Y_2] = a_1a_2 + a_1b_2E[{X_2}^{c_2}] + a_2b_1E[{X_1}^{c_1}] + b_1b_2E[{X_1}^{c_1}]E[{X_2}^{c_2}]$$

Calculating the numerator of the correlation equation:

$$\begin{aligned}\mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_2]\mathrm{E}[Y_2] &= a_1a_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right) + a_1b_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_2}^{c_2}]\\ &+ a_2b_1\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}] + b_1b_2\left(1+\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}]E[{X_2}^{c_2}]\\ &- (a_1a_2 + a_1b_2E[{X_2}^{c_2}] + a_2b_1E[{X_1}^{c_1}] + b_1b_2E[{X_1}^{c_1}]E[{X_2}^{c_2}])\end{aligned}$$

Cancelling terms:

$$\begin{aligned}\mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_2]\mathrm{E}[Y_2] &= a_1a_2\left(\cancel{1+}\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right) + a_1b_2\left(\cancel{1+}\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_2}^{c_2}]\\ &+ a_2b_1\left(\cancel{1+}\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}] + b_1b_2\left(\cancel{1+}\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}]E[{X_2}^{c_2}]\\ &- (\cancel{a_1a_2 + a_1b_2E[{X_2}^{c_2}] + a_2b_1E[{X_1}^{c_1}] + b_1b_2E[{X_1}^{c_1}]E[{X_2}^{c_2}]})\end{aligned}$$

$$\begin{aligned}\mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_2]\mathrm{E}[Y_2] &= a_1a_2\left(\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right) + a_1b_2\left(\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_2}^{c_2}]\\ &+ a_2b_1\left(\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}] + b_1b_2\left(\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)E[{X_1}^{c_1}]E[{X_2}^{c_2}]\end{aligned}$$

$$\begin{aligned}\mathrm{E}[Y_1Y_2] - \mathrm{E}[Y_2]\mathrm{E}[Y_2] &= \left(\rho_{\varepsilon_1,\varepsilon_2}\sigma_{\varepsilon_1}\sigma_{\varepsilon_2}\right)(a_1a_2 + a_1b_2E[{X_2}^{c_2}] + a_2b_1E[{X_1}^{c_1}]\\ &+ b_1b_2E[{X_1}^{c_1}]E[{X_2}^{c_2}])\end{aligned}$$

Finally, using Step 3 we arrive at:

$$\rho_{X,Y} = \frac{\left(\rho_{\varepsilon_1,\varepsilon_2}\right)(a_1a_2 + a_1b_2E[{X_2}^{c_2}] + a_2b_1E[{X_1}^{c_1}] + b_1b_2E[{X_1}^{c_1}]E[{X_2}^{c_2}])}{\prod b_{\mathrm{i}}\sqrt{[E^2({X_i}^{c_{\mathrm{i}}})Var(\varepsilon_i)] + [Var({X_i}^{c_{\mathrm{i}}})] + [Var({X_i}^{c_{\mathrm{i}}})Var(\varepsilon_i)]}}$$

Case 1: if $c_{\mathrm{i}} = 1$, then $Y_{\mathrm{i}} = (a_{\mathrm{i}} + b_{\mathrm{i}}x)\varepsilon_{\mathrm{i}}$

$$\rho_{X,Y} = \frac{\left(\rho_{\varepsilon_1,\varepsilon_2}\right)\left(a_1a_2 + a_1b_2\mu_{X_2} + a_2b_1\mu_{X_1} + b_1b_2\mu_{X_1}\mu_{X_2}\right)}{\prod b_{\mathrm{i}}\sqrt{[E^2(X_i)Var(\varepsilon_i)] + [Var(X_i)] + [Var(X_i)Var(\varepsilon_i)]}}$$

Case 2: if $a_{\mathrm{i}} = 0$, and $c_{\mathrm{i}} = 1$ then $Y_{\mathrm{i}} = b_{\mathrm{i}}x\varepsilon_{\mathrm{i}}$

$$\rho_{X,Y} = \frac{(\rho_{\varepsilon_1,\varepsilon_2})(b_1 b_2 \mu_{X_1}\mu_{X_2})}{\prod b_i \sqrt{[E^2(X_i)Var(\varepsilon_i)] + [Var(X_i)] + [Var(X_i)Var(\varepsilon_i)]}}$$

8.6 Type III-2 Functional Correlation

Type III-2 functional correlation exists between pairs of RVs that are related to each other through different functions of their dependent variables. One example of Type III-2 correlation is the correlation between two summary-level (parent) WBS elements that have correlated lower-level WBS elements (i.e., their children). The WBS shown in Table 8-6 has costs that are correlated with $\boldsymbol{\rho}$ (a correlation matrix).

Table 8-6 Example WBS

WBS	μ	σ
1.	37.000	10.325
1.1	10.000	4.000
1.2	12.000	5.000
1.3	15.000	6.000
2.	36.000	10.555
2.1	18.000	7.000
2.2	6.000	3.000
2.3	12.000	5.000

The matrix, $\boldsymbol{\rho}$, representing the correlation between each of the lower-level WBS elements is shown below.

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}$$

Using the values of σ_i of the lower-level WBS elements shown in Table 8-6, we are able to compute the standard deviations of summary-level WBS elements σ_1 , σ_2, and σ_{Tot}. The correlation matrix above can be partitioned into four sub-matrices, or partitions. The matrix shown in Figure 8-1 shows the partitions used to calculate σ_1 (upper left) and σ_2 (lower right). The remaining two partitions represent the correlation between WBS elements that are children of different parent WBS elements.

ρ i,j	1.1	1.2	1.3	2.1	2.2	2.3
1.1	1	0.2	0.2	0.2	0.2	0.2
1.2	0.2	1	0.2	0.2	0.2	0.2
1.3	0.2	0.2	1	0.2	0.2	0.2
2.1	0.2	0.2	0.2	1	0.2	0.2
2.2	0.2	0.2	0.2	0.2	1	0.2
2.3	0.2	0.2	0.2	0.2	0.2	1

Figure 8-1 Partitioned Correlation Matrix

The correlation coefficient between WBS elements 1 and 2 can be represented by $\rho_{1,2}$. This value is related to the lower left and upper right correlation coefficients in the partitioned correlation matrix.

Remembering that ${\sigma_{Tot}}^2 = \sum_{i=1}^{n} {\sigma_i}^2 + 2\sum_{k=j+1}^{n}\sum_{j=1}^{n-1} \rho_{j,k}\sigma_j\sigma_k$, we can express σ_{Tot} in two ways. The first uses the variances and covariance of the summary elements,

${\sigma_{Tot}}^2 = {\sigma_1}^2 + {\sigma_2}^2 + 2\rho_{1,2}\sigma_1\sigma_2$, and the second uses the variances and covariances of the lower-level WBS elements,

${\sigma_{Tot}}^2 = {\sigma_{1.1}}^2 + \cdots + {\sigma_{2.3}}^2 + 2\left(\rho_{1.1,1.2}\sigma_{1.1}\sigma_{1.2} + \cdots + \rho_{2.2,2.3}\sigma_{2.2}\sigma_{2.3}\right)$.

Since both equal σ_{Tot}, we can say

$${\sigma_1}^2 + {\sigma_2}^2 + 2\rho_{1,2}\sigma_1\sigma_2 = {\sigma_{1.1}}^2 + \cdots + {\sigma_{2.3}}^2 + 2\left(\rho_{1.1,1.2}\sigma_{1.1}\sigma_{1.2} + \cdots + \rho_{2.2,2.3}\sigma_{2.2}\sigma_{2.3}\right)$$

By solving for $\rho_{1,2}\sigma_1\sigma_2$, we get the correlation between WBS elements 1 and 2:

$$\rho_{1,2} = \frac{\left(\rho_{1.1,1.2}\sigma_{1.1}\sigma_{1.2} + \cdots + \rho_{2.2,2.3}\sigma_{2.2}\sigma_{2.3}\right) + \frac{1}{2}\left[\left({\sigma_{1.1}}^2 + \cdots + {\sigma_{2.3}}^2\right) - \left({\sigma_1}^2 + {\sigma_2}^2\right)\right]}{\sigma_1\sigma_2}.$$

8.6.1 Type III-2 Functional Correlation Example

For our example, we will continue the calculation with values from Table 8-6.

If we calculate σ_{Tot} using lower-level WBS elements we have ${\sigma_{Tot}}^2 = 160$ (or $\sigma_{Tot} = 17.550$).

Finding the terms for the formula used to calculate the correlation coefficient between WBS elements 1 and 2, we have:

$({\sigma_{1.1}}^2 + \cdots + {\sigma_{2.3}}^2) = 160$, and $({\sigma_1}^2 + {\sigma_2}^2) = 218$, so

$$\frac{({\sigma_{1.1}}^2 + \cdots + {\sigma_{2.3}}^2) - ({\sigma_1}^2 + {\sigma_2}^2)}{2} = \frac{(160 - 218)}{2} = -29,$$

$\left(\rho_{1.1,1.2}\sigma_{1.1}\sigma_{1.2} + \cdots + \rho_{2.2,2.3}\sigma_{2.2}\sigma_{2.3}\right) = 74$, and

$\rho_{1,2} = \frac{(74)+(-29)}{(10.325)(10.555)} = \frac{45}{108.974} = 0.4129$

Using this value, along with σ_1 and σ_2 we have ${\sigma_{Tot}}^2 = {\sigma_1}^2 + {\sigma_2}^2 + 2\rho_{1,2}\sigma_1\sigma_2$.

${\sigma_{Tot}}^2 = (10.325)^2 + (10.555)^2 + 2(0.4129)(10.325)(10.555) = 160$, or $\sigma_{Tot} = 17.550$.

8.7 Section Summary

Knowing how to compute functional correlations allows us to use MOM summation in a WBS structure and to solve many of the problems germane to probabilistic schedule network analysis. The functional correlation between elements of cost and schedule models allows the analyst to determine their influence on the total variance of an estimate and to construct joint probability density functions of pairs of modeled variables such as cost and schedule.

9 Discrete Risks

Analysts may need to include the probabilistic impacts of unique, independent, and discrete risk events in an estimate developed with a particular method (e.g., parametrically, with a CER) that does not account for their impacts in their underlying assumptions. We will define a set of individual risks, R_i, as independent events with respect to (WRT) each other. We will also assume each R_i has a probability of occurrence of P_i and an associated impact of D_i.[52] These unique, independent risks are denoted as $R_i(P_i, D_i)$.

The PMF for each R_i is:

$$f_{R_i}(x) = \begin{cases} P_i & ; x = D_i \\ 1 - P_i & ; otherwise \end{cases} \quad \textbf{9-1}$$

The PMF $f_{R_i}(x)$ has two possible values: one in which the risk occurs with probability, P_i, and one where no risk occurs with probability $1 - P_i$. This discrete risk has two possible *states*, or a set of potential outcomes. The problem becomes more interesting (and practical) when we are dealing with more than one risk. If we have n possible risks, where $n \geq 1$, we will have k risk states (possible outcomes) as defined by the binomial coefficient[53], S_i: $0 \leq i \leq k$, where:

$$k = \sum_{i=0}^{n} \binom{n}{i} = 2^n \quad \textbf{9-2}$$

When we add a single discrete risk (R_1) to the estimate (C), a new type of distribution called a *mixed distribution*[54] is formed from the continuous distribution of C and the discrete distribution of R_1 (Evans & Rosenthal, 2010).[55] The mixed distribution will have mean μ_M and standard deviation σ_M. The statistics of the mixed distribution are not well publicized in the cost analysis literature, so we will first introduce the formulae for μ_M and σ_M for the simple single-risk case, then the more difficult multiple-risk case, and finally the general formulae that treat the impacts of a discrete risk as random variables.

9.1.1 Single Discrete Risk Case

In this case, we have one discrete risk (R_1) and therefore two possible states defined by k, where $k = 2^1 = 2$. These states are: (1) $S_0 = \overline{R_1}$, where R_1 does not occur, and (2) $S_1 = R_1$, where R_1 does occur. This situation is depicted in the Venn diagram in Figure 9-1.

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[52] The impact, D_i, may be either a discrete or a random variable (with parameters μ_{D_i} and σ_{D_i}). When D_i is a random variable, the discrete risk R_i is actually a mixed distribution.
[53] By an "outcome," we mean a combination of the n possible risks composed of those that actually occur.
[54] The mixed distribution is also called a "mixture distribution".
[55] Evans, M. J., & Rosenthal, J. S. (2010). Probability and Statistics: The Science of Uncertainty, 2nd Ed. New York, NY: W. H. Freeman and Co.

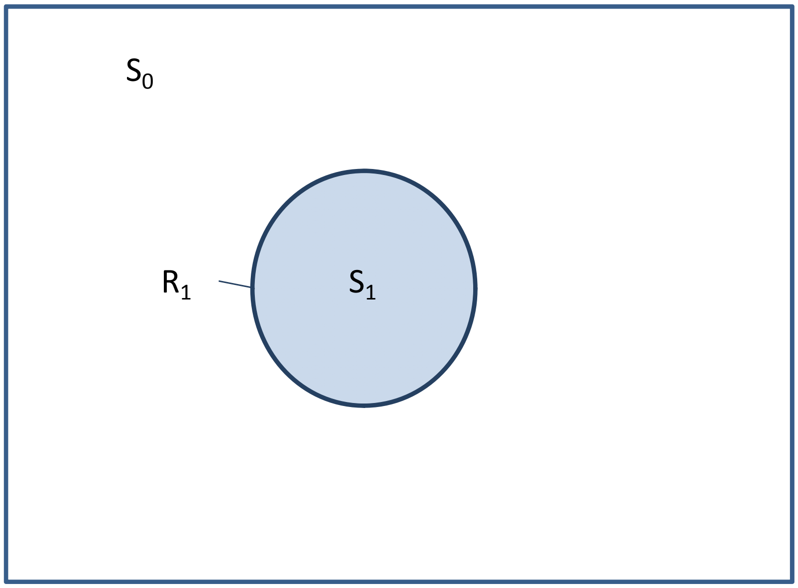

Figure 9-1 Venn Diagram Representation of Single-Risk State

If we use the same continuous distribution (C), and apply the discrete risk (R_1) with probability of occurrence P_1 and cost impact D_1, then this results in a *multimodal*, mixed probability distribution. This multimodal probability distribution will have $k = 2$ localized peaks or modes, defined by the number of possible states with the height of each mode defined by the probability of occurrence of the two states, S_0 and S_1 (Figure 9-2).

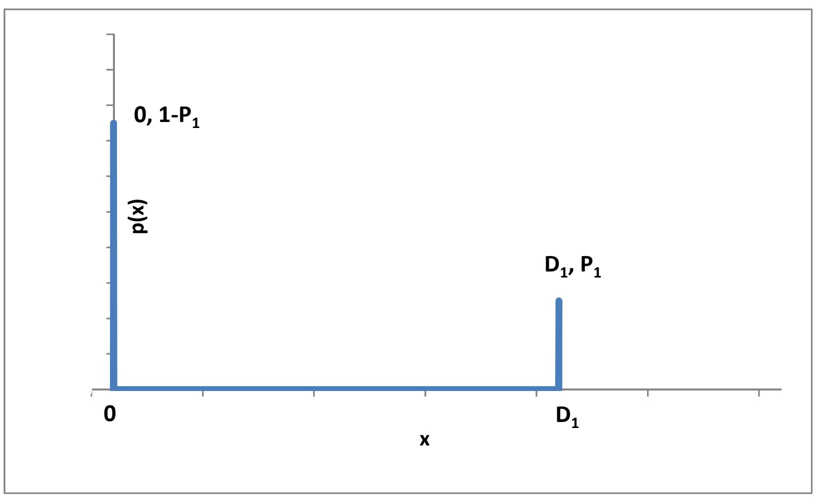

Figure 9-2 Probability Distribution of a Single Discrete Risk

When $P_1 = 0.5$, the probability of $S_0, P(S_0)$, is equal to the probability of $S_1, P(S_1)$. Since S_0 and S_1 have equal probabilities of occurrence, we expect the heights of the modes of the bimodal distribution to be equal, as shown in Figure 9-3, and the mean of the mixed distribution to be halfway between the two modes of the distribution.

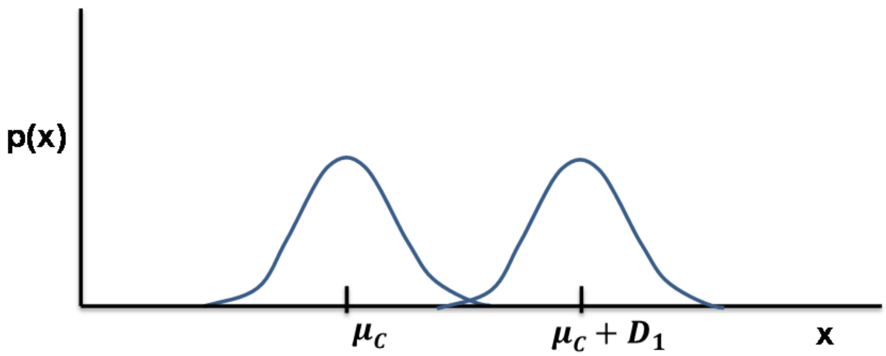

Figure 9-3 Normal Probability Distribution, $C_0 + R_1$, with $R_1(0.5, D_1)$

When $P_1 < 0.5$, the probability of S_0, $P(S_0)$, is greater than the probability of S_1, $P(S_1)$. Since S_0 has a greater probability of occurrence than S_1, we expect the height of the mode formed by S_0 to be greater than the mode formed by S_1 as shown in Figure 9-4. Additionally, the mean of the mixed distribution will be smaller than in the case of Figure 9-3.

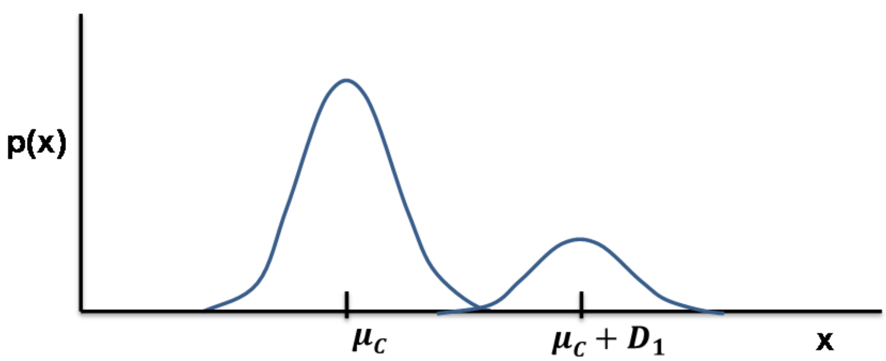

Figure 9-4 Normal Probability Distribution, C_0+D_1, with Low P_1

It is convenient to provide the information about the possible states, their probabilistic meaning, impact, and probabilities of occurrence in a state table such as the one shown in Table 9-1.

Table 9-1 Single Discrete Risk State Table

State, S_i	Definition	Risk Impact, D_{S_i}	Probability, $P(S_i)$
$S_0 = \overline{R_1}$	No risks occur	0	$[1 - P(R_1)]$
$S_0 = R_1$	R_1 occurs	D_1	$P(R_1)$

9.1.2 Mean of Mixed Distribution

The μ_M and σ_M of the mixed distributions will be weighted by the probabilities of occurrence of the two states, $P(S_0)$ and $P(S_1)$. μ_M is calculated using Equation 9-3.

$$\mu_M = \sum_{i=0}^{k} P(S_i)\mu_{S_i} = \sum_{i=0}^{k} P(S_i)\left(\mu_C + D_{S_i}\right) \qquad \textbf{9-3}$$

Equation 9-3 reduces to Equation 9-4 for any number of risks ($n; 1 \leq n$). This derivation is found in Appendix C – Derivations.

$$\mu_M = \mu_C + \sum_{j=1}^{n}(P_j D_j) \qquad \textbf{9-4}$$

Using Equation 9-3 for the single risk case, where there are two states, we can equate $P(S_0) = 1 - P(S_1)$. Using Equation 9-3, $P(S_0) = P_1$, and $P(S_1) = 1 - P_1$, so the mean of the mixed distribution formed by a single risk is $\mu_M = P(S_0)\mu_{S_0} + P(S_1)\mu_{S_1} = (1 - P_1)(\mu_C) + P_1\,(\mu_C + D_1) = \mu_C + P_1 D_1$. This is the same result obtained using Equation 9-4.

By rearranging terms, the mean of the continuous distribution (C) is shifted in the mixed distribution formed by the single risk case by $\mu_M - \mu_C = P_1 D_1$. Likewise Equation 9-4 can be easily manipulated to provide the mean shift ($\delta\mu$) in Equation 9-5.

$$\delta\mu = \mu_M - \mu_C = \sum_{j=1}^{n}(P_j D_j) \qquad \textbf{9-5}$$

9.1.3 Standard Deviation of Mixed Distribution

The standard deviation of the mixed distribution formed by n discrete risks and k states is the square root of the variance of the continuous distribution and the probability-weighted variances of the discrete risk states about μ_M :[56]

$$\sigma_M = \sqrt{(\sigma_C)^2 + \sum_{i=0}^{k-1} P(S_i)\left[D_{S_i} - (\mu_M - \mu_C)\right]^2}, \text{ so} \qquad \textbf{9-6}$$

$$\sigma_M = \sqrt{(\sigma_C)^2 + \sum_{i=0}^{k-1} P(S_i)\left[D_{S_i} - \delta\mu\right]^2}, \text{ where}$$

D_{S_i}= the impact of a particular state S_i

Expanding the summations in Equation 9-6 and using the relationship derived in Equation 9-5, we can derive a relationship for the standard deviation of the mixed distribution formed by C and a single discrete risk, R_1.

$$\delta\mu = \sum_{j=1}^{n}(P_j D_j) = P_1 D_1$$

$$\sigma_M = \sqrt{(\sigma_C)^2 + P(S_0)\left[D_{S_0} - \delta\mu\right]^2 + P(S_1)\left[D_{S_1} - \delta\mu\right]^2}$$

Using the expressions for $P(S_0)$, $P(S_1)$, D_{S_0}, and D_{S_1} from Table 9-1, we obtain

[56] This comes from the analogy of the variance of a distribution to the moment of inertia of an object with respect to an axis through the center of mass (the parallel axis theorem) from Ref 4: Helstrom, C.W., Probability and Stochastic Processes for Engineers, 2nd Ed, Macmillan, New York, 1991. p.113

$$\sigma_M = \sqrt{(\sigma_C)^2 + (1 - P_1)[0 - \delta\mu]^2 + (P_1)[D_1 - \delta\mu]^2}$$
$$\sigma_M = \sqrt{(\sigma_C)^2 + (1 - P_1)[-P_1 D_1]^2 + (P_1)[D_1 - P_1 D_1]^2}$$
$$\sigma_M = \sqrt{(\sigma_C)^2 + (1 - P_1)\left[{P_1}^2 {D_1}^2\right] + (P_1)\left[{D_1}^2 - 2P_1 {D_1}^2 + {P_1}^2 {D_1}^2\right]}$$
$$\sigma_M = \sqrt{(\sigma_C)^2 + \left[{P_1}^2 {D_1}^2\right] - P_1\left[{P_1}^2 {D_1}^2\right] + \left[P_1 {D_1}^2\right] - 2\left[{P_1}^2 {D_1}^2\right] + P_1\left[{P_1}^2 {D_1}^2\right]}$$
$$\sigma_M = \sqrt{(\sigma_C)^2 + \left[P_1 {D_1}^2\right] - \left[{P_1}^2 {D_1}^2\right]}$$

This simplifies to Equation 9-7.

$$\sigma_M = \sqrt{(\sigma_C)^2 + (1 - P_1)\left(P_1 {D_1}^2\right)} \qquad \textbf{9-7}$$

9.1.4 Multiple Risks Case

In the case where we have multiple risks, R_i, we have k possible states as defined by Equation 9-2. In the case where we have $n = 3$ risks, there will be $k = 2^3 = 8$ possible events as depicted in the Venn diagram (Rubenstein, 1986) in Figure 9-5.[57]

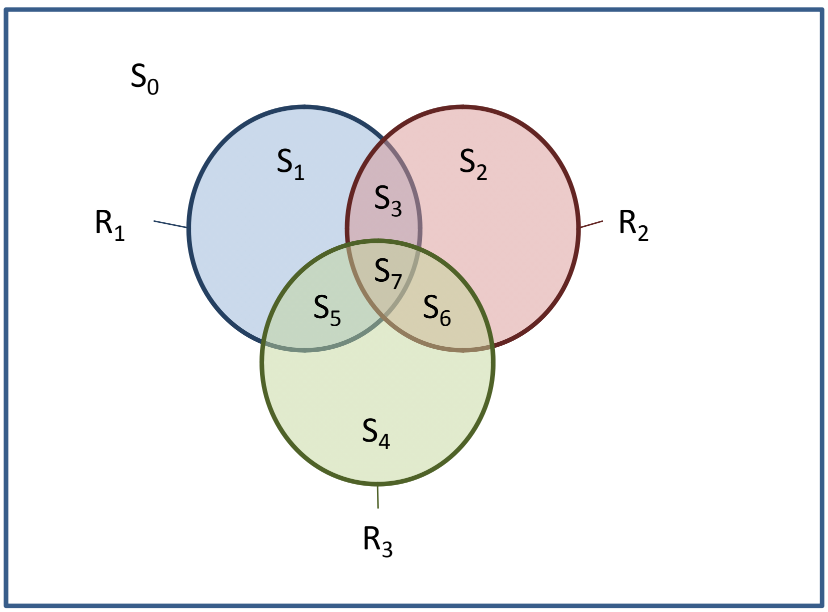

Figure 9-5 Venn Diagram Representation of Three-Risk State

Using the state table approach for the $n = 3$ risk case, we can list the $k = 8$ possible states, their probabilistic meanings, impacts, and probabilities of occurrence as shown in Table 9-2.

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[57] Rubenstein, M. F. (1986). Tools for Thinking and Problem Solving. Englewood Cliffs, NJ: Prentice-Hall.

Table 9-2 Multiple Discrete Risk State Table

State, S_i	Definition	Risk Impacts, D_{S_i}	Probability, P(S_i)
$S_0 = \overline{R_1} \cap \overline{R_2} \cap \overline{R_3}$	No risks occur	0	$[1-P_1][1-P_2][1-P_3]$
$S_1 = R_1 \cap \overline{R_2} \cap \overline{R_3}$	Only R_1 occurs	D_1	$P_1[1-P_2][1-P_3]$
$S_2 = \overline{R_1} \cap R_2 \cap \overline{R_3}$	Only R_2 occurs	D_2	$[1-P_1]P_2[1-P_3]$
$S_3 = R_1 \cap R_2 \cap \overline{R_3}$	R_1 and R_2occur	$D_1 + D_2$	$P_1P_2[1-P_3]$
$S_4 = \overline{R_1} \cap \overline{R_2} \cap R_3$	Only R_3 occurs	D_3	$[1-P_1][1-P_2]P_3$
$S_5 = R_1 \cap \overline{R_2} \cap R_3$	R_1 and R_3 occur	$D_1 + D_3$	$P_1[1-P_2]P_3$
$S_6 = \overline{R_1} \cap R_2 \cap R_3$	R_2 and R_3 occur	$D_2 + D_3$	$[1-P_1]P_2P_3$
$S_7 = R_1 \cap R_2 \cap R_3$	All risks occur	$D_1 + D_2 + D_3$	$P_1P_2P_3$

When the three discrete risks are combined probabilistically with the estimate (C), the result is a multimodal distribution with modes defined by the $k-1=7$ scaled copies of the continuous distribution (C). The scaling of each of these copies is weighted by that particular state's $P(S_i)$.

9.1.5 Multiple Discrete Risks Example

In the case where $n=3$, one possible distribution formed by the $k=8$ states where risks R_1, R_2, or R_3 are present is shown in Figure 9-6. The continuous distribution C is defined by a normal distribution, $N(1{,}0.2)$, and the three discrete risks are defined by $R_i(P_i, D_i)$: $R_1(0.4{,}1)$, $R_2(0.3, 2)$, and $R_3(0.2, 3)$.

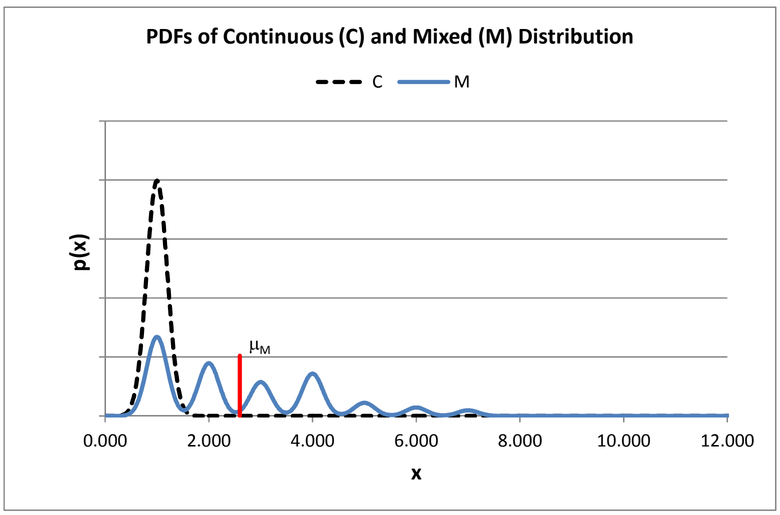

Figure 9-6 PDFs of Continuous (C) and Mixed Distributions

The mean of the mixed distribution is calculated from Equation 9-4 as

$$\mu_M = \mu_C + \sum_{j=1}^{n}(P_jD_j) = 1 + [(0.4)(1) + (0.3)(2) + (0.2)(3)] = 1 + [0.4 + 0.6 + 0.6] = 2.6.$$

The mean shift, $\delta\mu$, which is required to calculate σ_M is calculated using Equation 9-5 as

$\delta\mu = \mu_M - \mu_C = 2.6 - 1.0 = 1.6.$

The calculation of the standard deviation using Equation 9-6 requires calculation of the probability-weighted distribution of the means of the distributions formed by the k states, $P(S_i)\left[D_{S_i} - \delta\mu\right]^2$. These calculations are shown in Table 9-3.

Table 9-3 Three Discrete Risk Example Calculations

S_i	D_{S_i}	$(D_{S_i} - \delta\mu)^2$	$P(S_i)$	$P(S_i)(D_{S_i} - \delta\mu)^2$
S_0	0	$(-1.6)^2 = 2.56$	$[0.6][0.7][0.8] = 0.336$	$(2.56)(0.336) = 0.8602$
S_1	1	$(-0.6)^2 = 0.36$	$[0.4][0.7][0.8] = 0.224$	$(0.36)(0.224) = 0.0806$
S_2	2	$(0.4)^2 = 0.16$	$[0.6][0.3][0.8] = 0.144$	$(0.16)(0.144) = 0.0230$
S_3	3	$(1.4)^2 = 1.96$	$[0.4][0.3][0.8] = 0.096$	$(1.96)(0.096) = 0.1882$
S_4	3	$(1.4)^2 = 1.96$	$[0.6][0.7][0.2] = 0.084$	$(1.96)(0.084) = 0.1646$
S_5	4	$(2.4)^2 = 5.76$	$[0.4][0.7][0.2] = 0.056$	$(5.76)(0.056) = 0.3226$
S_6	5	$(3.4)^2 = 11.56$	$[0.6][0.3][0.2] = 0.036$	$(11.56)(0.036) = 0.4162$
S_7	6	$(4.4)^2 = 19.36$	$[0.4][0.3][0.2] = 0.024$	$(19.36)(0.024) = 0.4646$
$\sum P(S_i)(D_{S_i} - \delta\mu)^2 =$				**2.52**

Finally, we can calculate σ_M using Equation 9-6 as

$$\sigma_M = \sqrt{(\sigma_C)^2 + \sum_{i=0}^{k-1} P(S_i)\left[D_{S_i} - \delta\mu\right]^2} = \sqrt{(0.2)^2 + 2.52} = \sqrt{2.56} = 1.6.$$

The method of preparing state tables to perform the σ_M calculations becomes cumbersome when the number of discrete risks grows large, so we will develop formulae and introduce a software routine to ease the computational burden.

9.1.6 Binary State Representation

Since the number of expected states for these binomial events given n discrete risks is always 2^n, we can determine which risks occur in each state through binary representation of the state number S_0 to $S_{(2^n-1)}$. Conveniently, the binary representation of $k = 2^n$ states has n binary digits, or *bits*, corresponding to the number of risks. Since n binary digits represent 2^n unique combinations, we can uniquely determine which risks occur in any state S_0 to $S_{(2^n-1)}$. This is a fundamental application of the number of states of n binary switches, which is the foundation of Boolean addressing in computers (Kal, 2002).[58]

[58] Kal, S. (2002). Basic Electronics: Devices, Circuits and IT Fundamentals. New Delhi, India: Prentice Hall.

We will first define the rightmost digit as the first digit which indicates whether R_1 occurs in this state (1) or does not occur in this state (0). The digit to the left of the first digit is the second digit which indicates whether R_2 occurs or not, and the leftmost digit as the third, and so on. As an example, we will assume we have three risks ($n = 3$) and examine the third possible state, S_3. The state index, 3, is represented by the binary number (011). Since each of the binary digits represents whether a risk, R_i, occurs in S_3 we can determine: 1) digit one =1, so R_1 occurs in S_3; 2) digit two = 1, so R_2 occurs in S_3 ; and 3) digit three = 0, so R_3 does not occur in S_3 .

9.1.6.1 Bit Detection

The calculation of $P(S_i)$ in Table 9-2 benefits greatly from this method of bit detection.[59] We will define the bit indicator function $\gamma_{i,j}$ to represent the binary value of bit j of integer i. Using the example for S_3 above, we can detect the bits representing the risks R_1, R_2 and R_3 and determine which of the risks j occurs in S_3 . First, set $i = 3$ then $\gamma_{3,1} = 1$, $\gamma_{3,2} = 1$, and $\gamma_{3,3} = 0$.

We can express $P(S_i)$ in terms of $\gamma_{i,j}$ as

$$P(S_i) = \sum_{j=1}^{n} \big(1 - \gamma_{i,j}\big)\big(1 - P_j\big) + \gamma_{i,j} P_j. \qquad \textbf{9-8}$$

Similarly, we can use $\gamma_{i,j}$ to determine the impact of state i, D_{S_i} as

$$D_{S_i} = \sum_{j=1}^{n} \gamma_{i,j} D_j \; . \qquad \textbf{9-9}$$

Equations 9-8 and 9-9 greatly simplify the problem of calculating $P(S_i)$, D_{S_i}, and σ_M.

9.1.7 Adding Discrete Risks with Impacts that are Random Variables

Until now, we have discussed the situation of discrete risks having discrete impact. Since the risk impacts are also estimates (and contain some uncertainty), we can modify Equations 9-4 and 9-6 to accommodate risk impacts that are random variables.

Replacing the discrete value for D_j in Equation 9-4 with the mean of μ_{D_j}, we re-define μ_M to be

$$\mu_M = \mu_C + \sum_{j=1}^{n} \left(P_j \, \mu_{D_j}\right) \qquad \textbf{9-10}$$

This remains relatively unchanged as does Equation 9-5, which now intuitively becomes

$$\delta\mu = \mu_M - \mu_C = \sum_{j=1}^{n} \left(P_j \mu_{D_j}\right). \qquad \textbf{9-11}$$

[59] The number of risks we can detect will be limited by the largest integer we are able to compute and find the binary equivalent.

The calculation of σ_M becomes more complicated by the fact that the impacts of the discrete risks are random variables. Remembering the equation for the variance of the sum of distributions in Equation 4-4, we must treat the variance of the sum of the continuous distribution (C) and the risk impacts (D_j) at any particular state in the same fashion. Using linear algebra (Covert, 2006), we can rewrite Equation 4-4 in matrix form as

$$\sigma_T{}^2 = \boldsymbol{\sigma}^{\boldsymbol{T}}\boldsymbol{\rho}\boldsymbol{\sigma}, \text{ where} \qquad \textbf{9-12}$$

$\boldsymbol{\sigma}$ is a column vector of standard deviations with dimension *1*x*M*, and
$\boldsymbol{\rho}$ is the correlation matrix with dimension *M*x*M*.

We will use this convenient expression for calculating the impacts of the variances of each D_j on σ_m . To begin, for each state S_i , we must compose a (partitioned) vector of standard deviations ($\boldsymbol{\sigma_i}$) of dimension *1*x*M*. Since we will be calculating the variance of the statistical sum of C and n risks, the number of rows will be $M = n + 1$. The top row element is σ_C, and the remaining n rows are the products of $\sigma_{D_j}\gamma_{i,j}$ representing the binary detection multiplied by the standard deviation of the risk impact as shown in Equation 9-13.

$$\boldsymbol{\sigma_i} = \begin{bmatrix} \underline{\sigma_C} \\ \sigma_{D_1}\gamma_{i,1} \\ \vdots \\ \sigma_{D_n}\gamma_{i,n} \end{bmatrix} \qquad \textbf{9-13}$$

Next, we must compose the correlation matrix ($\boldsymbol{\rho}$) of dimension *M*x*M*

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho_{C,D_1} & \cdots & \rho_{C,D_n} \\ \rho_{C,D_1} & 1 & \cdots & \rho_{D_1,D_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{C,D_n} & \rho_{D_1,D_n} & \cdots & 1 \end{bmatrix} \qquad \textbf{9-14}$$

Using the form of Equation 9-12, we calculate the probability-weighted variance $\sigma_{D_{S_i}}{}^2$ for each state as shown in Equation 9-15.

$$\sigma_{D_{S_i}}{}^2 = \left(P_{S_i}\boldsymbol{\sigma_i}\right)^{\boldsymbol{T}}\boldsymbol{\rho}\boldsymbol{\sigma_i} \qquad \textbf{9-15}$$

Finally, σ_M is computed by taking the square root of the two components that determine the variance of the mixed distribution: 1) the variance of the sum of the probability-weighted variances (Equation 9-15), and 2) the probability-weighted distribution of the means of the distributions formed by the k states Equations 9-6 and 9-11.

$$\sigma_M = \sqrt{\sum_{i=0}^{2^n-1} P(S_i)\left(\sigma_{D_{S_i}}\right)^2 + \left\{P(S_i)\left[D_{S_i} - \sum_{j=1}^{n}\left(P_j \mu_{D_j}\right)\right]^2\right\}} \qquad \textbf{9-16}$$

$$\sigma_M = \sqrt{\sum_{i=0}^{2^n-1} P(S_i)\left\{\left(\sigma_{D_{S_i}}\right)^2 + \left[D_{S_i} - \delta\mu\right]^2\right\}}$$

Equation 9-16 reduces to Equation 9-6 when all $\sigma_{D_\mathrm{i}} = 0$ and $\mu_{D_j} = D_j$.

9.1.8 Discrete Risk Numerical Example

As a demonstration, we will use the multiple discrete risks example shown previously, except each D_i will be defined by a normal distribution $N[\mu, \sigma]$ with the parameters shown in Table 9-4, and (C) defined as a "risk" with 100% probability of occurrence. We will use a constant value of $\rho = 0.2$ between all random variables.

Table 9-4 Three Discrete Risk Example Inputs

	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	$\boldsymbol{P_i}$
$\boldsymbol{C}$	1.0	0.2	1.0
$\boldsymbol{R_1}$	1.0	0.2	0.4
$\boldsymbol{R_2}$	2.0	0.3	0.3
$\boldsymbol{R_3}$	3.0	0.6	0.2

Using the example inputs, we can easily calculate μ_M using Equation 9-10 as follows:

$$\mu_M = 1.0 + (0.4)(1.0) + (0.3)(2.0) + (0.2)(3.0) = 2.6$$

Next, we calculate $\delta\mu$ using Equation 9-11

$$\delta\mu = \sum_{j=1}^{n}\left(P_j \mu_{D_j}\right) = (0.4)(1.0) + (0.3)(2.0) + (0.2)(3.0) = 1.6$$

Then calculate ${\sigma_{D_{S_i}}}^2$ using 9-15 and $\rho = 0.2$ and ${\sigma_{D_{S_i}}}^2 = \sum_{i=0}^{2^n-1}\left(P_{S_i}\boldsymbol{\sigma_i}\right)^{\boldsymbol{T}}\boldsymbol{\rho\sigma_i} = 0.1892$.

Finally, we combine the terms in Equation 9-16

$$\sigma_M = \sqrt{\sum_{i=0}^{2^n-1} P(S_i)\left\{\left(\sigma_{D_{S_i}}\right)^2 + \left[D_{S_i} - \delta\mu\right]^2\right\}} = 1.6460$$

To check this result, a 100,000-trial statistical simulation using Crystal Ball ® using the same inputs for the example shown above provided the following results:

Exact (Eq. 4-48 & 4-54)	**Simulated**
$\mu_M = 2.6000$	$\hat{\mu}_M = 2.6004$
$\sigma_M = 1.6460$	$\hat{\sigma}_M = 1.6495$

The difference between the simulated results and the calculated results is due to the statistical simulation's inability to exactly sample perfectly-distributed correlated random variables. We can extract the 100,000 samples and determine 1) the correlation of the samples used in the simulation, and 2) the frequency of S_i. Using this information, we can re-calculate μ_M and σ_M to see the effect of sampling error from the simulation.

Table 9-5 shows the Pearson correlation of the statistical samples. Note the correlation coefficients between different R_i (shaded on left) were defined to be $\rho = 0.2$ but are slightly different in the simulation samples. Also, the different independent risk probabilities P_i (shaded on right) were specified to be uncorrelated probabilities of occurrence, but do not have $\rho = 0.0$. Additionally, there is spurious correlation between the PDF of the risks (C and R_i) and the probabilities of occurrence of the risks (in italics).

Table 9-5 Correlation of Samples from Statistical Simulation

	C	R_1	R_2	R_3	P_1	P_2	P_3
C	1.0000	0.2015	0.2048	0.2002	*-0.0007*	*0.0008*	*-0.0015*
$\mathbf{R_1}$		1.0000	0.2111	0.2074	*0.0005*	*0.0011*	*0.0039*
$\mathbf{R_2}$			1.0000	0.2077	*-0.0038*	*0.0030*	*0.0055*
$\mathbf{R_3}$				1.0000	*0.0011*	*0.0024*	*-0.0002*
$\mathbf{P_1}$					1.0000	0.0020	0.0026
$\mathbf{P_2}$						1.0000	0.0017
$\mathbf{P_3}$							1.0000

Since the risks can no longer be assumed to be independent, we can extract the state probabilities $P(S_i)$, which are provided in Table 9-6.

Table 9-6 State probabilities $P(S_i)$ from Statistical Simulation

S_i	$P(S_i)$
0	0.33706
1	0.22326
2	0.14345
3	0.09623
4	0.0834
5	0.05628
6	0.03609
7	0.02423

We can substitute the sampled values from the simulation (ρ from Table 9-5 and $P(S_i)$ from Table 9-6) into Equations 9-10 and 9-16. This results in calculations for the mean and standard deviation of the mixed distribution much closer to the simulated values.

Exact (Eq. 4-48 & 4-54)	**Simulated**	**Exact Using ρ and $P(S_i)$ from Simulation**
$\mu_M = 2.6000$	$\hat{\mu}_M = 2.6004$	$\mu_M = 2.6000$
$\sigma_M = 1.6460$	$\hat{\sigma}_M = 1.6495$	$\sigma_M = 1.6489$

The evidence that the statistical simulation cannot exactly sample perfectly-distributed correlated random variables shows the equations developed in this report are more reliable calculators of discrete risk than are simulated results.

10 Maximum and Minimum of Random Variables

The maximum duration of the paths of a schedule network define its critical path, and in a probabilistic schedule, the distributions of the probabilistic critical paths define the probabilistic schedule duration. If the tasks in a schedule network are defined by probability distributions (i.e., PDFs or PMFs), we may need to find the moments and the distribution of the maximum of two or more probability distributions where these tasks merge. If the finish date of a schedule is defined by the latest end date of three tasks, A, B, and C, which is defined by $max(A, B, C)$. This is equivalent to $max(max(A, B), C)$ and $max(A, max(B, C)$, which is an important consideration because it allows us to deal with the problem of finding the moments of the maximum of distributions in pairs.

The random variable representing the maximum of two correlated distributions X_1 and X_2 can be defined as the function $V = max\{X_1, X_2\}$. To find the PDF of V, we must first find its CDF and differentiate to find the PDF. In the independent case,

$$F_V(v) = F_{X_1}(v)F_{X_2}(v). \qquad \textbf{10-1}$$

To find the PDF we take the derivative WRT. v:

$$f_V(v) = f_{X_1}(v)F_{X_2}(v) + F_{X_1}(v)f_{X_2}(v). \qquad \textbf{10-2}$$

The correlated case is much harder to solve. Fortunately, *IEEE Transactions on Very Large Scale Integration (VLSI) Systems* (Nadarajah & Kotz, 2008) provides a method of calculating the first two moments of the max and min of two correlated Gaussian distributions.[60, 61]

The PDF of $X=max(X_1,X_2)$ is $f(x) = f_1(x) + f_2(x)$, where **10-3**

$$f_1(x) = \frac{1}{\sigma_1}\varphi\left(\frac{\mu_1-x}{\sigma_1}\right)\Phi\left(\frac{\rho_{1,2}(\mu_1-x)}{\sigma_1\sqrt{1-\rho_{1,2}{}^2}} - \frac{(\mu_2-x)}{\sigma_2\sqrt{1-\rho_{1,2}{}^2}}\right)$$

$$f_2(x) = \frac{1}{\sigma_2}\varphi\left(\frac{\mu_2-x}{\sigma_2}\right)\Phi\left(\frac{\rho_{1,2}(\mu_2-x)}{\sigma_2\sqrt{1-\rho_{1,2}{}^2}} - \frac{(\mu_1-x)}{\sigma_1\sqrt{1-\rho_{1,2}{}^2}}\right)$$

Where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the PDF and the CDF of the standard normal distribution, respectively.

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[60] Nadarajah, S., & Kotz, S. (2008, Feb.). Exact Distribution of the Max/Min of Two Gaussian Random Variables. IEEE Transactions on VLSI Systems, 16(2), 210-212.

[61] The integrated circuit industry has a deep interest in scheduling methods and routines which stems from the need to calculate signal transit and arrival times at nodes in integrated circuit paths.

The first two moments of $X = max(X_1, X_2)$ are

$$E[X] = \mu_1 \Phi\left(\frac{\mu_1-\mu_2}{\theta}\right) + \mu_2 \Phi\left(\frac{\mu_2-\mu_1}{\theta}\right) + \theta\varphi\left(\frac{\mu_1-\mu_2}{\theta}\right) \qquad \textbf{10-4}$$

$$E[X^2] = (\sigma_1^2 + \mu_1^2)\Phi\left(\frac{\mu_1-\mu_2}{\theta}\right) + (\sigma_2^2 + \mu_2^2)\Phi\left(\frac{\mu_2-\mu_1}{\theta}\right) + (\mu_1 + \mu_2)\theta\varphi\left(\frac{\mu_1-\mu_2}{\theta}\right) \qquad \textbf{10-5}$$

$$\theta = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} \qquad \textbf{10-6}$$

where $\rho_{1,2}$ = Pearson correlation between tasks X_1 and X_2, and

$$\sigma^2 = E[(X-\mu)^2] = E[X^2] - \mu^2 \qquad \textbf{10-7}$$

The moments of the maximum and minimum of two joint lognormal distributions have been published (Lien, 2005) and are useful when dealing with maximums of sums of random variables that exhibit lognormal behavior.[62] The first two raw moments of the bivariate lognormal distribution are provided in 10-8 and 10-9.

$$E[X] = \mu_1 \Phi\left[\frac{(P_1-P_2)+(Q_1{}^2-\rho Q_1 Q_2)}{\theta}\right] + \mu_2 \Phi\left[\frac{(P_2-P_1)+(Q_2{}^2-\rho Q_1 Q_2)}{\theta}\right] \qquad \textbf{10-8}$$

$$E[X^2] = (\sigma_1^2 + \mu_1^2)\Phi\left(\frac{P_1-P_2}{\theta}\right) + (\sigma_2^2 + \mu_2^2)\Phi\left(\frac{P_2-P_1}{\theta}\right) \qquad \textbf{10-9}$$

$\theta = \sqrt{Q_1^2 + Q_2^2 - 2\rho Q_1 Q_2}$ where the correlation between their underlying normal distributions is **10-10**

$\rho = \frac{1}{Q_1 Q_2} ln\left[1 + \rho_{1,2}\left(\sqrt{\left[e^{Q_1^2} - 1\right]\left[e^{Q_2^2} - 1\right]}\right)\right]$, and

$\rho_{1,2}$ = Pearson correlation between lognormal distributions of tasks X_1 and X_2

P_1, P_2, Q_1,and Q_2 are parameters of the lognormal distribution defined in Equations 4-5 and 4-6.

While these are useful expressions for calculating the moments of Gaussian distributions that are either user-defined or formed through the statistical summation of PDFs of serial tasks, they do not provide a solution to the problem of finding moments of the maximum of two non-Gaussian distributions (e.g., uniform or triangular). Fortunately, the moments of distributions in which we are interested represent the finish dates of tasks, and since these are often based on sums of durations of several tasks, we can assume the sum to be

[62] Lien, D. (2005). On the Minimum and Maximum of Bivariate Lognormal Random Variables. Extremes, 8, 79-83.

Gaussian. For completeness, we do need a method of working with the order statistics of non-Gaussian PDFs.

10.1.1 Maximum and Minimum of Correlated Non-Gaussian PDFs

The applied probability and statistics literature provides little insight into finding either the maximum or minimum of correlated non-Gaussian distributions. So, when we are dealing with correlated non-Gaussian distributions, the task is more difficult. For instance, when we are interested in the PDF of the maximum (or minimum) of two uniform distributions we have to go back to the fundamentals and derive a solution. Figure 10-1 provides examples of pairs of uniform distributions $U_1(L_1, H_1)$ and $U_2(L_2, H_2)$ that represent cases in which the maximum of these two distributions will be different.

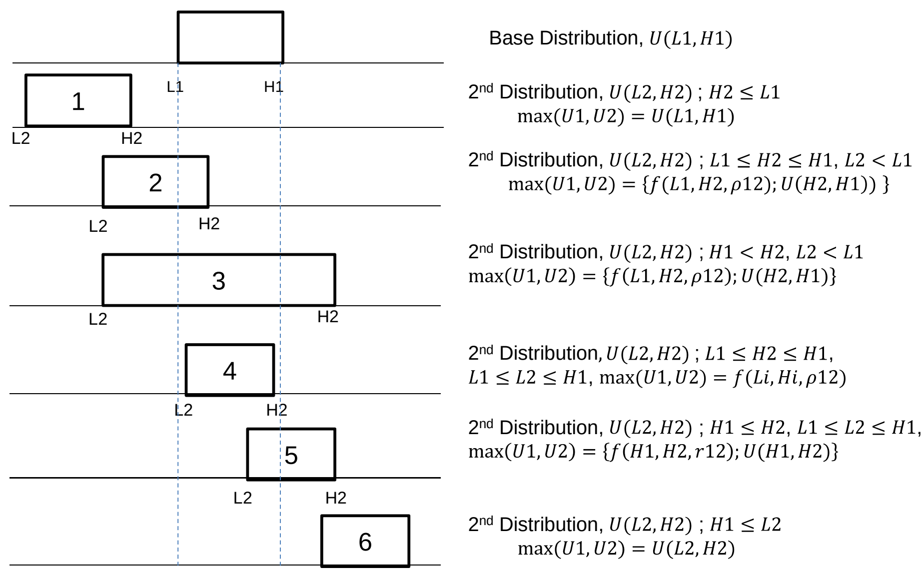

Figure 10-1 Pairs of Uniform Distributions with Varying Ranges

To find the PDF of the maximum of two distributions, we first must define a random variable, $V = max\{X_1, X_2\}$, where $X_1 = U(L_1, H_1)$, and $X_1 = U(L_2, H_2)$. We find the PDF of V by first finding its CDF, $F_V(v)$.

$$F_V(v) = P\{V \leq v) = P\{X_1 \leq v, X_2 \leq v\} \qquad \textbf{10-11}$$

In the independent case, $F_V(v) = F_X(v)F_Y(v)$. Now take the derivative with respect to v to get

$$f_V(v) = f_X(v)F_Y(v) + f_Y(v)F_X(v) \qquad \textbf{10-12}$$

The k^{th} moments are:

$$E[f_V{}^k(v)] = \int_{-\infty}^{\infty} v^k\, f_V(v)dv \qquad \textbf{10-13}$$

From which we can find the mean,

$$\mu = E[f_V(v)] \quad \textbf{10-14}$$

and standard deviation of the resulting distribution.

$$\sigma^2 = E[{f_V}^2(v)] - \mu^2 \quad \textbf{10-15}$$

The distributions of the maximums of the pairs of uniform distributions defined in Figure 10-1 are shown in Figure 10-2.

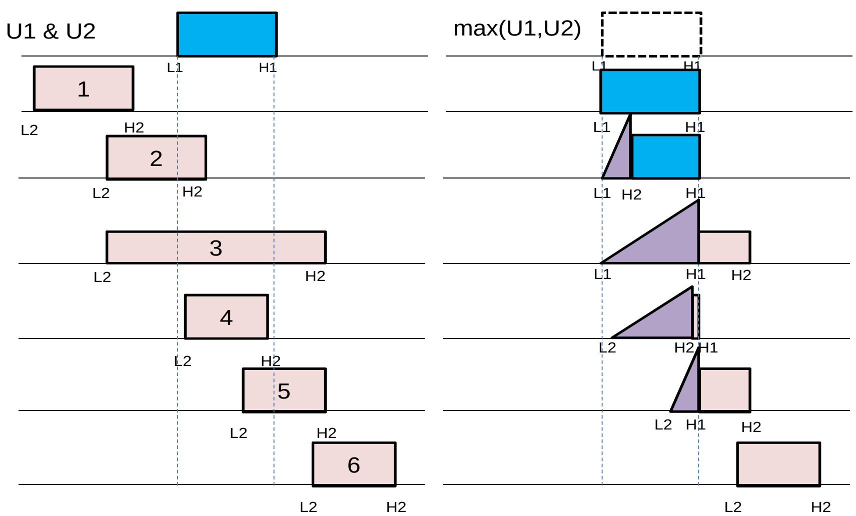

Figure 10-2 Maximum of Pairs of Uniform Distributions with Varying Ranges

In the correlated case, the Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions may provide a solution. The formula for the joint CDF is

$W(x,y) = F(x)G(y)\{1 + a[1 - F(x)][1 - G(y)]\}$, where the marginal PDFs $H(x,\infty) = F(x)$ and $G(\infty,y) = G(y)$ **10-16**

Unfortunately it can only model a limited range of Pearson correlations[63]; $-\frac{1}{3} < \rho < \frac{1}{3}$.

When $\rho_{X,Y} = 1$, the two distributions covary in the same direction with respect to (wrt) their means. When $\rho_{X,Y} = -1$, they covary in opposite directions wrt their means. When $\rho_{X,Y} \neq 0,\ and\ -1 < \rho_{X,Y} < 1$ the results are rather interesting.

[63] Schucany, W.R., Parr, W. C., and Boyer, J.E., (1978). Correlation Structure in Farlie-Gumbel-Morgenstern Distributions. Biometrika, 65(3), 650-653.

We will show some statistical simulation results to illustrate the effects of correlation on the maximum of two uniform distributions in the following figures. We assume U_1 [1,5], U_2 [1,3], and $\rho = \{-1.0, -0.9, -0.5,\ 0,\ 0.5, 0.9, 1.0\}$.

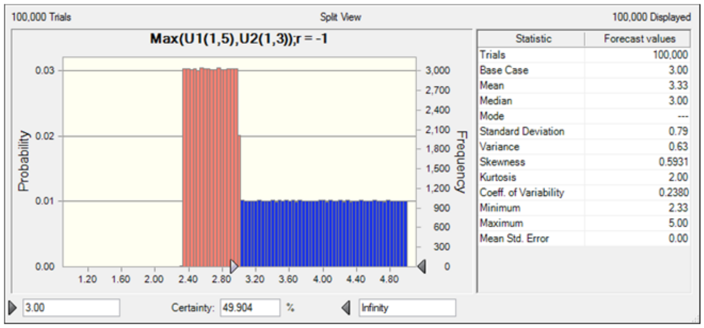

Figure 10-3 Max of U_1 and U_2 where ρ = -1.0

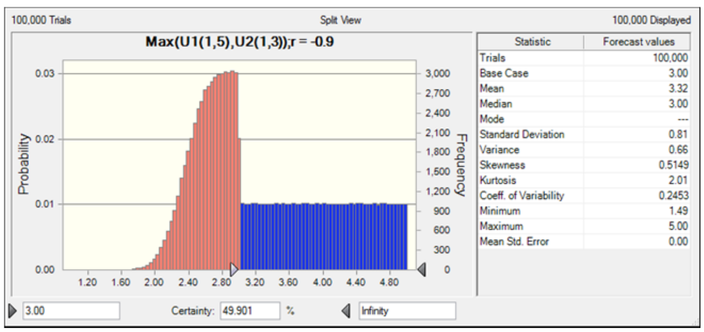

Figure 10-4 Max of U_1 and U_2 where ρ = -0.9

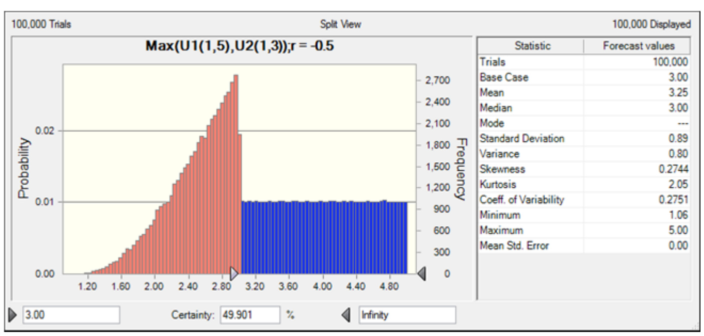

Figure 10-5 Max of U_1 and U_2 where ρ = -0.5

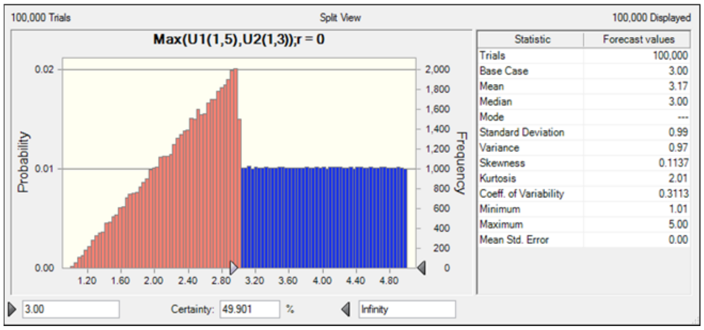

Figure 10-6 Max of U_1 and U_2 where $\rho = 0.0$

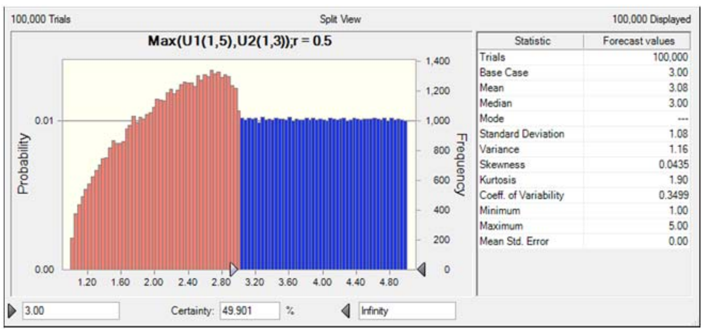

Figure 10-7 Max of U_1 and U_2 where $\rho = 0.5$

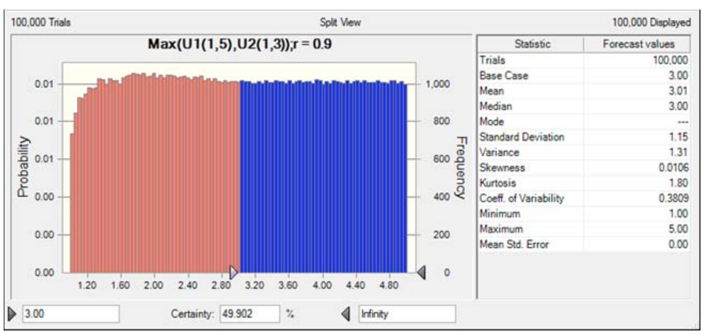

Figure 10-8 Max of U_1 and U_2 where $\rho = 0.9$

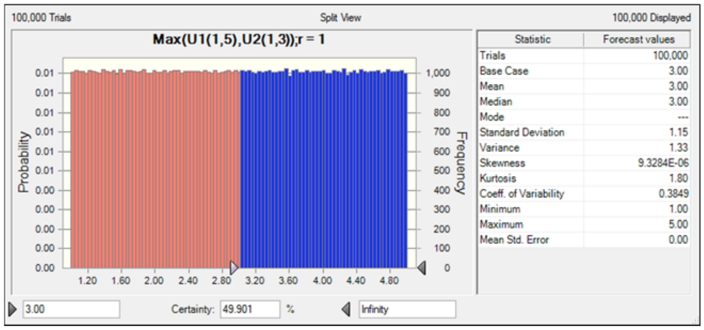

Figure 10-9 Max of U_1 and U_2 where ρ = 1.0

The PDF of the maximum of the two distributions modeled by a FGM, where $\rho = \alpha/3$, and $-1 \leq \alpha \leq 1$) is:

$$\max(U_1, U_2) = h(u) = (1+\alpha)[F(u)g(u) + f(u)G(u)] + \alpha\{f(u)G(u)[2F(u)G(u) - 2F(u) - G(u)] + F(u)g(u)[2F(u)G(u) - 2G(u) - F(u)]\}$$

A plot of this function is shown in Figure 10-10.

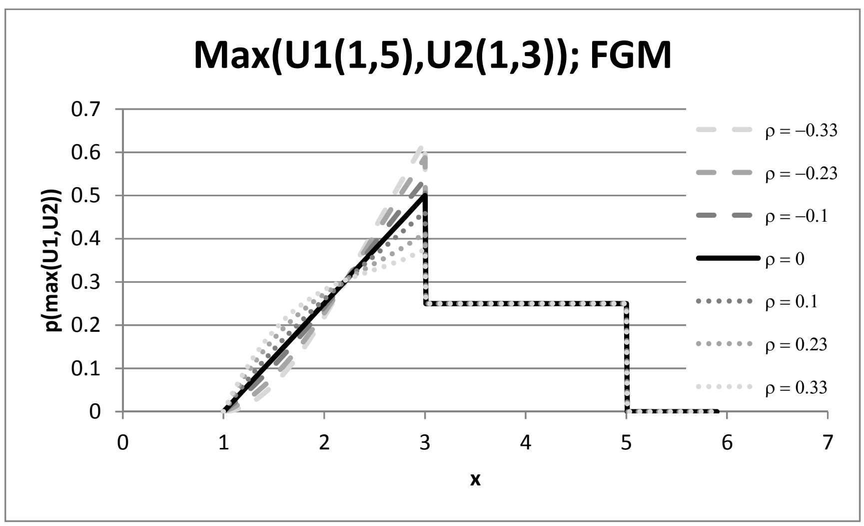

Figure 10-10 Max of U_1 and U_2 using FGM Copula

Further work needs to be done to increase the effective use of FGM copulas to find the maximum of two correlated, non-Gaussian distributions.

11 Example Problems

To demonstrate the techniques presented in previous sections, we will perform analytic uncertainty and risk assessments on a parametric estimating model and a resource-loaded schedule model – both resulting in a joint PDF of cost and schedule. Each example will model the cost risk, the schedule risk and the joint cost and schedule risk.

11.1 Parametric Estimate Example Problem

The model chosen for the parametric example is an estimate of the cost and schedule used to explain functional correlation in Section 8. The schedule duration is estimated using a series of fictitious schedule estimating relationships (SERs). The joint probability distribution of cost and schedule is formed using the marginal distributions of cost and schedule. We will demonstrate the formation of these three distributions and compare their statistics with those generated from a 100,000-trial statistical simulation.

11.1.1 Cost Distribution

To calculate the marginal distribution of the cost of the system, we follow the FRISK method described in Section 4.2.2. In the first step of the FRISK method, we define the mathematical problem to be solved – which is defining the WBS of the system and the CERs. We will reuse the WBS and CERs defined in Section 8 and repeat them in Table 11-1. In the second step of the FRISK method, we define the probability distributions of the inputs (also shown in Table 11-1), and their correlations.

Table 11-1 Level 1 WBS Elements for Parametric Example

	WBS Element, i	CER, i	Drivers	X_i	ε_i
1	Systems Engineering, Program Management Integration and Test (SEITPM)	$Y_1 = 0.498{X_1}^{0.9}\varepsilon_1$	PMP	$\sim L\left(\sum_{i=2}^{10}\mu_i\ ,\ \sqrt{\boldsymbol{\sigma}^T\boldsymbol{\rho}\boldsymbol{\sigma}}\right)$	L(1,0.49)
	Prime Mission Product (PMP)	$\sum_{i=2}^{10} Y_i$	Sum of Hardware and Software costs		0
2	Antenna	$Y_2 = 34.36{X_{2a}}^{0.5}{X_{2b}}^{0.8}\varepsilon_2$	Aperture Diameter (m), Frequency (GHz)	T(2,3,4) T(16,17,18)	L(1,0.30)
3	Electronics	$Y_3 = 30.06{X_3}^{0.8}\varepsilon_3$	Frequency (GHz)	T(16,17,18)	L(1,0.40)
4	Platform	$Y_4 = 26.91{X_{4a}}^{0.5}{X_{4b}}^{0.85}\varepsilon_4$	Aperture Diameter (m), Number of Axes	T(2,3,4) Constant = 2	L(1,0.38)
5	Facilities	$Y_5 = 1.64{X_5}^{0.8}\varepsilon_5$	Area (m^2)	T(18,20,22)	L(1,0.25)
6	Power Distribution	$Y_6 = 0.32{X_6}^{0.9}\varepsilon_6$	Electrical Power (W)	T(1200,1425,1875)	L(1,0.18)
7	Computers	$Y_7 = 0.58{X_7}^{0.87}\varepsilon_7$	MFLOPS	T(180,200,220)	L(1,0.31)
8	Environmental Control	$Y_8 = 1.94{X_8}^{0.4}\varepsilon_8$	Heat Load (W)	T(1100,1200,1300)	L(1,0.21)
9	Communications	$Y_9 = 5.62{X_9}^{0.9}\varepsilon_9$	Data Rate (MBPS)	T(25,30,35)	L(1,0.28)
10	Software	$Y_{10} = 1.38{X_{10}}^{1.2}\varepsilon_{10}$	Effective Source Lines of Code, eKSLOC	T(80,90,130)	L(1,0.32)

There are no correlations between different technical parameters used as inputs to the CERs in this example, and there are no correlations between the error of CER 1 and any errors of the other CERs, but the correlations between the errors of CERs 2 through 10 are set to 0.2 ($\rho_{\varepsilon_i,\varepsilon_j=0.2}$; $\forall i \geq 2$). The correlation matrix of the errors is shown in Figure 11-1.

$\rho_{\varepsilon i,\varepsilon j}$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
3	0	0.2	1	0.2	0.2	0.2	0.2	0.2	0.2	0.2
4	0	0.2	0.2	1	0.2	0.2	0.2	0.2	0.2	0.2
5	0	0.2	0.2	0.2	1	0.2	0.2	0.2	0.2	0.2
6	0	0.2	0.2	0.2	0.2	1	0.2	0.2	0.2	0.2
7	0	0.2	0.2	0.2	0.2	0.2	1	0.2	0.2	0.2
8	0	0.2	0.2	0.2	0.2	0.2	0.2	1	0.2	0.2
9	0	0.2	0.2	0.2	0.2	0.2	0.2	0.2	1	0.2
10	0	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	1

Figure 11-1 Correlations between Errors of CERs 1 through 10

There is a mix of different types and orders of functional correlation in this example problem as shown in Figure 11-2. CER 1 is functionally correlated to the other CERs through its use of PMP as its cost driver (a Type I-2 correlation). CERs 2 and 3 and CERs 2 and 4 are correlated through the reuse of a cost driver (a Type II-1 correlation). The remaining CER pairs are correlated to each other through their correlated multiplicative errors (a Type III-1 correlation).

$\rho_{yi,yj}$	1	2	3	4	5	6	7	8	9	10
1	1.0000	I-2	I-2	I-2	I-2	I-2	I-2	I-2	I-2	I-2
2	I-2	1.0000	II-1	II-1	III-1	III-1	III-1	III-1	III-1	III-1
3	I-2	II-1	1.0000	III-1	III-1	III-1	III-1	III-1	III-1	III-1
4	I-2	II-1	III-1	1.0000	III-1	III-1	III-1	III-1	III-1	III-1
5	I-2	III-1	III-1	III-1	1.0000	III-1	III-1	III-1	III-1	III-1
6	I-2	III-1	III-1	III-1	III-1	1.0000	III-1	III-1	III-1	III-1
7	I-2	III-1	III-1	III-1	III-1	III-1	1.0000	III-1	III-1	III-1
8	I-2	III-1	III-1	III-1	III-1	III-1	III-1	1.0000	III-1	III-1
9	I-2	III-1	III-1	III-1	III-1	III-1	III-1	III-1	1.0000	III-1
10	I-2	III-1	III-1	III-1	III-1	III-1	III-1	III-1	III-1	1.0000

Figure 11-2 Types of Functional Correlation in Example Problem

11.1.2 Probability Distributions

The third step of the FRISK method is the calculation of the means and variances of the WBS element costs. The first WBS element, SEITPM, is a cost-on-cost CER of the PMP (i.e., the sum of the individual estimates of WBS elements 2 through 10). The remaining WBS elements are estimated using non-cost-driven CERs. Because the first WBS element relies on the cost estimates of the other WBS elements, we must first calculate the means and variances of the costs of WBS elements 2 through 10 (i.e., PMP cost) then use those results to calculate the mean and variance of the first WBS element.

The moments of the estimates from non-cost-driven CERs are calculated using the propagation of errors method demonstrated in Section 7. As an example, WBS element 6 is estimated using the following CER from Section 8:

$$Y_6 = 0.32{X_6}^{0.9}\varepsilon_6$$

$\mathrm{E}[\mathrm{X}_6] = \mu_{\mathrm{X}_6}$, which is found using Equation 4-1.

Since X_6 is defined by the triangular PDF, T(1200,1425,1875),

$$\mu_{\mathrm{X}_6} = \frac{1200+1425+1875}{3} = 1500$$

$\mathrm{E}[Y_6]$ can be found by using expectation methods or Mellin transforms. In this example, we will use expectation methods to compute $\mathrm{E}[Y_6]$.

$\mathrm{E}[Y_6] = \mathrm{E}\left[0.32{X_6}^{0.9}\varepsilon_6\right] = 0.32\mathrm{E}\left[{X_6}^{0.9}\right]\mathrm{E}[\varepsilon_6]$, and since $\mathrm{E}[\varepsilon_6] = 1$, $\mathrm{E}[Y_6] = 0.32\mathrm{E}\left[{X_6}^{0.9}\right]$.

Since X_6 is a triangular PDF, we must find the expectation of a triangular PDF raised to a power, which is:

$$E\left[X^{\mathrm{k}}\right] = \frac{2}{(H-L)(M-L)}\left\{\frac{M^{k+2}-L^{k+2}}{k+2} - L\,\frac{M^{k+1}-L^{k+1}}{k+1}\right\} + \frac{2}{(H-L)(H-M)}\left\{H\,\frac{H^{k+1}-M^{k+1}}{k+1} - \frac{H^{k+2}-M^{k+2}}{k+2}\right\}$$

Substituting the parameters L, M, H and k using our example, $\mathrm{E}\left[{X_6}^{0.9}\right] = 721.626$, so $\mathrm{E}[Y_6] = (0.32)(721.626) = 230.920$.

$Var(X_6)$ is calculated using the square of one half of the population standard deviation of the distributions parameters. This equates to:

$Var(X_6) = \left(\frac{STDEVP(1200,1425,1875)}{2}\right)^2 = 19687.5$, so $\sigma_{X_6} = \sqrt{19687.5} = 140.31$

The variance of Y is calculated using the propagation of errors method, since the CER, f_{Y_6}, and its error are independent RVs.

$Var(Y) = \left(\mu_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2 + \left(\sigma_{f_{Y_6}}\right)^2 + \left(\sigma_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2$; where

$\sigma_{\varepsilon_6} = 0.18$ (from Table 11-1), and $\mu_{f_{Y_6}} = 230.920$ (found using functional correlation Step 2a)

$\sigma_{f_{Y_6}}$ is found using the equation for the transformation of a triangular PDF from Section 4.3.3.

$$\sigma_{f_{Y_6}} = b\sqrt{\frac{2}{(H-L)}\left[\frac{1}{(M-L)}\left\{\frac{M^{2c+2}-L^{2c+2}}{2c+2} - L\frac{M^{2c+1}-L^{2c+1}}{2c+1}\right\} + \frac{1}{(H-M)}\left\{H\frac{H^{2c+1}-M^{2c+1}}{2c+1} - \frac{H^{2c+2}-M^{2c+2}}{2c+2}\right\}\right] - \left(\frac{\mu_f}{b}\right)^2}$$

By substituting the coefficient $b = 0.32$ and the triangular distribution parameters, L, M and H into this equation we get $\sigma_{f_{Y_6}} = 19.428$.

So $\sigma_{Y_6} = \sqrt{\left(\mu_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2 + \left(\sigma_{f_{Y_6}}\right)^2 + \left(\sigma_{f_{Y_6}}\sigma_{\varepsilon_6}\right)^2}$

$$\sigma_{Y_6} = \sqrt{[(230.920)(0.18)]^2 + [19.428]^2 + [(19.428)(0.18)]^2} = 46.015$$

The remaining moments of the cost estimates of the non-cost-driven CERs in the example problem are computed in a similar manner and are shown in Table 11-2. The means and standard deviations of the analytic results match closely with the results obtained using the 100,000-trial statistical simulation. The results of the analytic method and the statistical simulation are a close match.

Table 11-2 Moments of WBS Elements with Non-Cost-Driven CERs

WBS #	Analytic		Simulation	
	μ	σ	μ	σ
2	572.706	177.022	572.676	176.900
3	289.953	116.136	289.962	116.172
4	83.829	32.484	83.824	32.463
5	18.014	4.544	18.014	4.543
6	230.920	46.015	230.911	45.977
7	58.248	18.186	58.244	18.172
8	33.068	6.960	33.068	6.959
9	119.965	34.446	119.962	34.420
10	347.121	120.764	347.121	120.787

The PMP cost is the sum of WBS elements 2 through 10, so its mean is $\mu_{PMP} = \sum_2^{10} \mu_i$ and its standard deviation is calculated through the linear algebraic relationship, $\sigma_{PMP} = \sqrt{\boldsymbol{\sigma^T \rho \sigma}}$. μ_{PMP} is simple to compute and is $\mu_{PMP} = \sum_2^{10} \mu_i = 1753.825$. The calculation of σ_{PMP} requires we know the correlation between pairs of CERs from 2 through 10, $\boldsymbol{\rho}$, which is the functional correlation sub-matrix between the elements of PMP.

Functional Correlation Matrix

The functional correlation matrix shown in Figure 11-2 contains a combination of Type I-2, II-1 and III-1 functional correlations. We use the examples provided in Section 8 of this report to develop these correlations, which are shown in Figure 11-3.

$\rho_{yi,yj}$	1	2	3	4	5	6	7	8	9	10
1	1.0000	0.2614	0.2098	0.1454	0.1156	0.1426	0.1273	0.1184	0.1393	0.2085
2	0.2614	1.0000	0.1969	0.2306	0.1924	0.1753	0.1927	0.1937	0.1893	0.1785
3	0.2098	0.1969	1.0000	0.1959	0.1979	0.1804	0.1983	0.1993	0.1948	0.1837
4	0.1454	0.2306	0.1959	1.0000	0.1944	0.1772	0.1947	0.1957	0.1912	0.1804
5	0.1156	0.1924	0.1979	0.1944	1.0000	0.1790	0.1968	0.1978	0.1933	0.1823
6	0.1426	0.1753	0.1804	0.1772	0.1790	1.0000	0.1794	0.1803	0.1762	0.1662
7	0.1273	0.1927	0.1983	0.1947	0.1968	0.1794	1.0000	0.1981	0.1936	0.1827
8	0.1184	0.1937	0.1993	0.1957	0.1978	0.1803	0.1981	1.0000	0.1946	0.1836
9	0.1393	0.1893	0.1948	0.1912	0.1933	0.1762	0.1936	0.1946	1.0000	0.1794
10	0.2085	0.1785	0.1837	0.1804	0.1823	0.1662	0.1827	0.1836	0.1794	1.0000

Figure 11-3 Functional Correlation Matrix for Example Problem

Using the functional correlation sub-matrix (i.e., the lower-right 9x9 elements of the matrix shown in Figure 11-3) and the sigmas of WBS elements 2 through 10, we can compute $\sigma_{PMP} = \sqrt{\boldsymbol{\sigma}^T \boldsymbol{\rho} \boldsymbol{\sigma}} = 331.917$. Now that we know the moments of PMP and the functional correlation sub-matrix, we can calculate the moments of the first WBS element, μ_{Y_1} and σ_{Y_1}. The results of this example calculation are shown in Section 8 and are repeated in Table 11-3. The results of the analytic method and the statistical simulation are a close match.

Table 11-3 Moments of WBS Elements

WBS #	Analytic		Simulation	
	μ	σ	μ	σ
1	413.170	201.048	413.090	200.916
2	572.706	177.022	572.676	176.900
3	289.953	116.136	289.962	116.172
4	83.829	32.484	83.824	32.463
5	18.014	4.544	18.014	4.543
6	230.920	46.015	230.911	45.977
7	58.248	18.186	58.244	18.172
8	33.068	6.960	33.068	6.959
9	119.965	34.446	119.962	34.420
10	347.121	120.764	347.121	120.787

Now that the necessary calculations to compute the moments of the total program cost are completed, the total cost mean, μ_Y, and the total cost sigma, σ_Y can be calculated.

$\mu_Y = \sum_1^{10} \mu_i$ and $\sigma_Y = \sqrt{\boldsymbol{\sigma}^T \boldsymbol{\rho} \boldsymbol{\sigma}}$, where $\boldsymbol{\sigma}$ is the vector of the sigmas of all of the WBS elements (Table x-3), and $\boldsymbol{\rho}$ is the full functional correlation matrix shown in Figure 11-3. The results of these calculations are shown in Table 11-4 along with the total mean and standard deviation obtained using the 100,000-trial statistical simulation. Again, the results are a close match.

Table 11-4 Moments of Total Program Cost

	Analytic		Simulation	
	μ	σ	μ	σ
Total	2166.995	443.915	2166.873	443.511

The total program cost is represented as a lognormal distribution and its parameters P_Y and Q_Y are calculated using Equations 4-5 and 4-6. The results are:

$P_Y = 7.452$, and $Q_Y = 0.188$.

Using these values, we can compute the percentiles of total cost, which are presented in Table 11-5.

Table 11-5 Table Percentiles of Total Cost

Percentile	Total Cost, Y
10%	1637.140582
20%	1789.878287
30%	1908.780462
40%	2016.616222
50%	2122.909227
60%	2234.804788
70%	2361.059157
80%	2517.905056
90%	2752.814045

The plot of the CDF of total cost is shown in Figure 11-4. Note that the original point estimate calculated using the modes of the triangular inputs shown in Table 11-1 is at the 48th percentile.

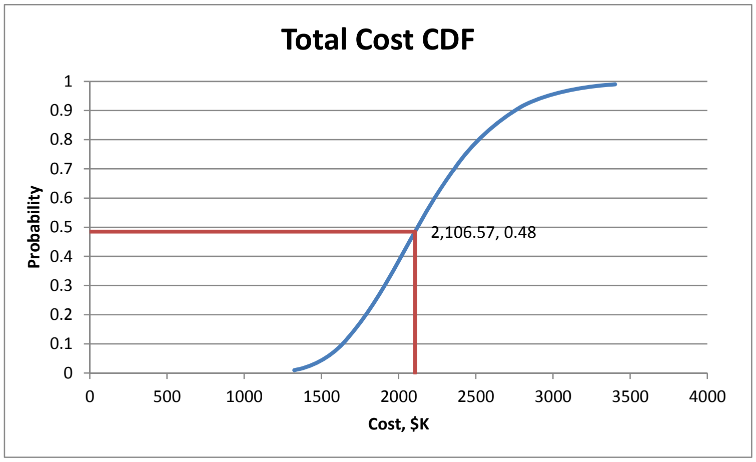

Figure 11-4 CDF of Total Cost, Y

11.1.2.1 Contribution to Variance

The contribution to the variance (CTV) shows which WBS elements most strongly influence the variance of total cost. The CTV of any WBS element, i, can be calculated using row i of the functional correlation matrix as follows:

$CTV_i = \sigma_i\,(\boldsymbol{\rho_i}\boldsymbol{\sigma})/{\sigma_Y}^2$, where
σ_i = the standard deviation of WBS element i
$\boldsymbol{\rho_i}$ = row i of the full functional correlation matrix (a vector)
$\boldsymbol{\sigma}$ = the vector of standard deviations of the WBS elements
${\sigma_Y}^2$ = total cost variance

The CTV of each of the WBS elements is shown in Figure 11-5.

WBS	CTV
1 SEITPM	32%
2 Antenna	27%
3 Electronics	14%
4 Platform	3%
5 Facilities	0%
6 Power Distribution	4%
7 Computers	1%
8 Environmental Control	0%
9 Communications	3%
10 Software	15%
Sum	1.0000

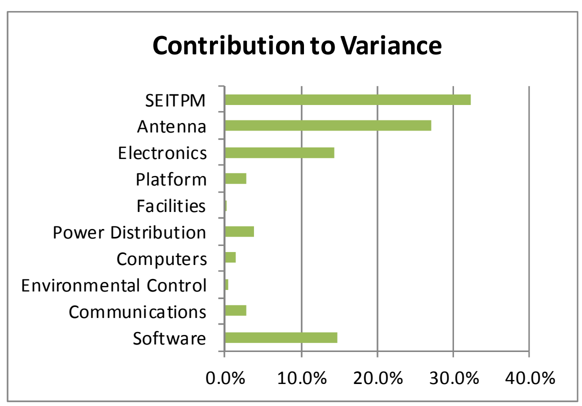

Figure 11-5 WBS Element Contribution to Variance

11.1.3 Schedule Probability Distribution

The program schedule is calculated using a fictitious schedule estimating relationship (SER) defined as the number of months from the authority-to-proceed (ATP) to the end of installation and checkout, $D = 0.21X_D{}^{1.2}\varepsilon_D$. The SER is similar to the Software CER and reuses its driver (effective source lines of code). The multiplicative error of the SER, ε_D, is defined by the lognormal distribution L(1,0.45). Since the SER in this example problem is similar to the CER of WBS element 10 (Software cost), we substitute the coefficients and multiplicative error distribution to directly calculate the moments of the resulting schedule distribution, which are:

$\mu_D = 52.823$, and $\sigma_D = 24.935$.

This distribution is assumed to be lognormal and has parameters $P_D = 3.866$ and $Q_D = 0.449$. A plot of the CDF of schedule duration is shown in Figure 11-6.

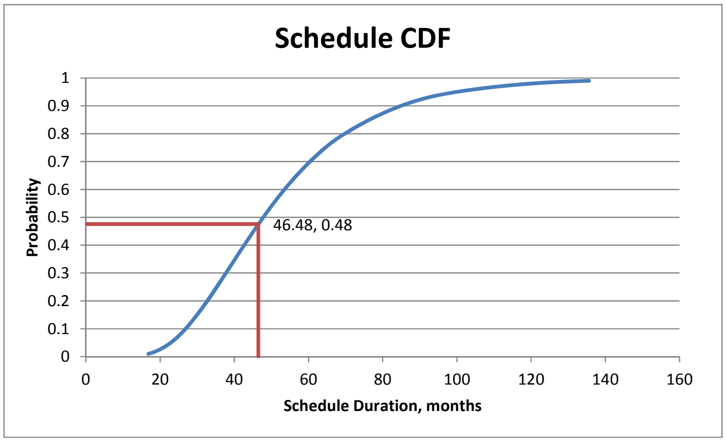

Figure 11-6 Schedule Duration CDF

11.1.4 Forming the Joint Distribution

The joint distribution of cost and schedule duration is formed using the marginal cost and schedule duration distributions in a bivariate lognormal distribution. This joint PDF is defined in Garvey (2000) as:[64]

$$BiL\left((P_1, P_2), (Q_1, Q_2, \rho_{1,2})\right) = f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi Q_1 Q_2 \sqrt{1-\rho_{1,2}^2}\, x_1 x_2} e^{-\left\{\frac{1}{2}w\right\}} \text{; where}$$

$$w = \frac{1}{1-\rho_{1,2}^2}\left[\left(\frac{ln(x_1)-P_1}{Q_1}\right)^2 - 2\rho_{1,2}\left(\frac{ln(x_1)-P_1}{Q_1}\right)\left(\frac{ln(x_2)-P_2}{Q_2}\right) + \left(\frac{ln(x_2)-P_2}{Q_2}\right)^2\right],$$

$$\rho_{1,2} = \frac{1}{Q_1 Q_2}\ln\left[1 + \rho_{X_1,X_2}\sqrt{e^{Q_1{}^2} - 1}\sqrt{e^{Q_2{}^2} - 1}\right], \text{ and}$$

ρ_{X_1,X_2} is the correlation coefficient between RVs X_1 and X_2.

[64] Garvey, P. R. (2000). Probability Methods for Cost Uncertainty Analysis: A Systems Engineeering Perspective. New York, NY: Marcel Dekker.

The correlation between cost and schedule is a Type II-2 functional correlation since cost (Y) and schedule duration (D) are nested functions of a common input, the effective source lines of code.

The Type II-2 functional correlation between cost and schedule duration is defined as:

$$\rho_{Y,D} = \frac{E[YD] - \mu_Y \mu_D}{\sigma_Y \sigma_D}.$$

The moments of Y and D have been previously calculated, however $E[YD]$ must be found. By expanding the product, YD, we get:

$YD = (\sum_{i=1}^{10} Y_i)\left(0.21{X_D}^{1.2}\varepsilon_D\right)$, which expands further to $YD = (\sum_{i=1}^{10} Y_i)\left(0.21{X_D}^{1.2}\varepsilon_D\right)$.

A fuller expansion of these terms is necessary to calculate the expectation. Substituting the equations of CERs 1 and 10 and setting $X_D = X_{10}$, we get:

$$YD = \left(0.498[\sum_{i=2}^{10} Y_i]^{0.9}\varepsilon_1 + 1.38{X_{10}}^{1.2}\varepsilon_{10} + \sum_{i=2}^{9} Y_i\right)\left(0.21{X_{10}}^{1.2}\varepsilon_D\right)$$

Through distribution of the SER, we get:

$$YD = (0.498[\sum_{i=2}^{10} Y_i]^{0.9}\varepsilon_1)\left(0.21{X_D}^{1.2}\varepsilon_D\right) + \left(1.38{X_{10}}^{1.2}\varepsilon_{10}\right)\left(0.21{X_{10}}^{1.2}\varepsilon_D\right) + (\sum_{i=2}^{9} Y_i)\left(0.21{X_{10}}^{1.2}\varepsilon_D\right)$$

Combining constants and similar variables results in:

$$YD = 0.1046[\sum_{i=2}^{10} Y_i]^{0.9}{X_{10}}^{1.2}\varepsilon_1\varepsilon_D + 0.2898{X_{10}}^{2.4}\varepsilon_{10}\varepsilon_D + (\sum_{i=2}^{9} Y_i)\left(0.21{X_{10}}^{1.2}\varepsilon_D\right)$$

Moving ${X_D}^{1.2}$ into the summation, we get a sum with three major terms:

$$YD = 0.1046\left[\sum_{i=2}^{10} Y_i\, X_{10}^{\frac{1.2}{0.9}}\right]^{0.9} \varepsilon_1\varepsilon_D + 0.2898{X_{10}}^{2.4}\varepsilon_{10}\varepsilon_D + \sum_{i=2}^{9} Y_i\left(0.21{X_D}^{1.2}\varepsilon_D\right)$$

11.1.4.1 Expectation of YD

The expectation of YD is:

$$\mathrm{E}[YD] = E\left[0.1046\left[\sum_{i=2}^{10} Y_i\, X_{10}^{\frac{1.2}{0.9}}\right]^{0.9}\right]E[\varepsilon_1]E[\varepsilon_D] + E\left[0.2898{X_{10}}^{2.4}\right]E[\varepsilon_{10}]E[\varepsilon_D] + E[\sum_{i=2}^{9} Y_i\, D].$$

From the first term, we can break up the X_{10} component and eliminate $E[\varepsilon_1]E[\varepsilon_D]$, since they are uncorrelated (i.e., $E[\varepsilon_1]E[\varepsilon_D] = 1$).

$$E\left[0.1046\left[\sum_{i=2}^{10} Y_i\, X_{10}^{\frac{1.2}{0.9}}\right]^{0.9}\right] = 0.1046E\left[1.38X_{10}^{\left\{1.2+\frac{1.2}{0.9}\right\}}\varepsilon_{10} + \sum_{i=2}^{9} Y_i\, X_D^{\frac{1.2}{0.9}}\right]^{0.9}$$

First Term of $E[YD]$

For convenience, we rename the first term $(1.38X_{10}{}^{\left\{1.2+\frac{1.2}{0.9}\right\}}\varepsilon_{10} + \sum_{i=2}^{9} Y_i\, X_D{}^{\frac{1.2}{0.9}})$ to $(V = V_1 + V_2)$, which results in:

$V_1 = 1.38X_{10}{}^{\left\{1.2+\frac{1.2}{0.9}\right\}}\varepsilon_{10}$, which, by inspection, is a lognormal distribution.

Computing V_1

The lognormal parameters of V_1(i.e., $P_{V_1}$and Q_{V_1}) can be computed as follows:

1) Compute the moments and lognormal parameters of $A = X_{10}{}^{\left\{1.2+\frac{1.2}{0.9}\right\}}$:
   a. $E[A] = \left[X_{10}{}^{\left\{1.2+\frac{1.2}{0.9}\right\}}\right] = E\left[X_{10}{}^{2.5333}\right] = 119237.5843$, and
   b. $Var(A) = \left(X_{10}{}^{2.5333}\right) = 1115733687$, so
   c. $P_A = 11.6511$, $Q_A = 0.2749$
   d. Propagate errors due to ε_{10}, where $E[\varepsilon_{10}] = 1$, $\sigma_{\varepsilon_{10}} = 0.32$
2) $V_1 = 1.38A\varepsilon_{10}$, so the moments and lognormal parameters are:
   a. $\mu_{V_1} = 1.38\mu_A E[\varepsilon_{10}] = 164547.866$
   b. $\sigma_{V_1} = 1.38\sqrt{\sigma_A{}^2 + \mu_A{}^2\sigma_{\varepsilon_{10}}{}^2 + \sigma_A{}^2\sigma_{\varepsilon_{10}}{}^2} = 68491.075$
   c. $P_{V_1} = 11.931$
   d. $Q_{V_1} = 0.400$

Computing V_2

The lognormal parameters of $V_2 = \sum_{i=2}^{9} Y_i X_D{}^{\frac{1.2}{0.9}}$ are able to be computed as well. First, we must compute the mean and sigma of $X_D{}^{\frac{1.2}{0.9}}$.

1) The variable X_D is a triangular distribution, so $X_D{}^{\frac{1.2}{0.9}}$ has the following moments:
   a. $\mu_{X_D{}^{\frac{1.2}{0.9}}} = 465.351$, $\sigma_{X_D{}^{\frac{1.2}{0.9}}} = 67.365$,
   b. $P_{X_D{}^{\frac{1.2}{0.9}}} = 6.132$, and $Q_{X_D{}^{\frac{1.2}{0.9}}} = 0.144$
2) For each WBS element from 2 to 9, compute the moments of $Y_i X_D{}^{\frac{1.2}{0.9}}$
   a. $P_{Y_i X_D{}^{\frac{1.2}{0.9}}\varepsilon_1{}^{\frac{1}{0.9}}} = P_{Y_i} + P_{X_D{}^{\frac{1.2}{0.9}}\varepsilon_1{}^{\frac{1}{0.9}}}$
   b. $Q_{Y_i X_D{}^{\frac{1.2}{0.9}}\varepsilon_1{}^{\frac{1}{0.9}}} = \sqrt{\left(Q_{Y_i}\right)^2 + \left(Q_{X_D{}^{\frac{1.2}{0.9}}\varepsilon_1{}^{\frac{1}{0.9}}}\right)^2}$ (in this case X_D and CERs 2 to 9 are independent)

The results of these calculations for WBS elements 2 through 9 are shown in Table 11-6.

Table 11-6 Moments of First Term, Part V_2

i	P_{Y_i}	Q_{Y_i}	$P_{Y_i X_D^{\frac{1.2}{0.9}}}$	$Q_{Y_i X_D^{\frac{1.2}{0.9}}}$	$\mu_{Y_i X_D^{\frac{1.2}{0.9}}}$	$\sigma_{Y_i X_D^{\frac{1.2}{0.9}}}$
2	6.305	0.302	12.437	0.335	266509.704	89209.682
3	5.595	0.386	11.728	0.412	134930.196	55589.229
4	4.359	0.374	10.491	0.401	39009.722	15643.639
5	2.860	0.248	8.993	0.287	8382.836	2407.118
6	5.423	0.197	11.555	0.244	107459.119	26253.462
7	4.018	0.305	10.151	0.337	27105.923	9144.805
8	3.477	0.208	9.609	0.253	15388.452	3895.777
9	4.748	0.281	10.880	0.316	55825.914	17654.212
Sum					654611.865	139002.227*

*This is not the sum of the individual sigmas.

μ_{V_2} is the sum of the means in Table 11-6. σ_{V_2} is calculated using $\sigma_{V_2} = \sqrt{\boldsymbol{\sigma}_{V_2}{}^T \boldsymbol{\rho} \boldsymbol{\sigma}_{V_2}}$, where $\boldsymbol{\rho}$ is the functional correlation sub-matrix of WBS elements 2 through 9 in Figure 11-3.

$\mu_{V_1} = 1.38\mu_A = 164547.866$, and $\mu_{V_2} = 654611.865$ (from Table 11-6), so μ_V is:

$\mu_V = \mu_{V_1} + \mu_{V_2} = 164547.866 + 654611.865 = 819159.732.$

$\sigma_{V_1} = 68491.075$, and $\sigma_{V_2} = 139002.227$ (from Table 11-6), so σ_V is the square root of the sum of the variances of V_1 and V_2:

$\sigma_V = \sqrt{(\sigma_{V_1}) + (\sigma_{V_2})} = 154960.1446.$

From μ_V and σ_V, we calculate the lognormal parameters P_V and Q_V using Equations 4-5 and4-6: $P_V = 13.598$, and $Q_V = 0.189$. The mean of the first term is computed by finding the expectation of an exponentiated lognormal RV:

$E[V^{0.9}] = 209577.473$, and by multiplication with the constant, 0.1046, we get:

$0.1046E[V^{0.9}] = 21918.22.$

Second Term of $E[YD]$

The second term is simple to compute, as:

$$E\left[0.2898{X_{10}}^{2.4}\varepsilon_{10}\varepsilon_D\right] = 0.2898E\left[{X_{10}}^{2.4}\right]E[\varepsilon_{10}\varepsilon_D] = 0.2898E\left[{X_{10}}^{2.4}\right]$$

Since X_{10} is triangularly distributed, with T(80,90,130), $E\left[{X_{10}}^{2.4}\right] = 64340.222$. The second term is $0.2898\mathrm{E}[64340.222] = 18645.796$.

The third term reduces to the following, since there are no correlated terms:

$$E\left[\sum_{i=2}^{9} Y_i\, D\right] = \sum_{i=2}^{9} E[Y_i D] = \sum_{i=2}^{9} E[Y_i]E[D] = \mu_D \sum_{i=2}^{9} \mu_{Y_i} = (52.823)(1406.704) = 74305.896$$

Summing these three terms gets us:

$$\mathrm{E}[YD] = E\left[0.1046\left[\sum_{i=2}^{10} Y_i\, {X_{10}}^{\frac{1.2}{0.9}}(\varepsilon_1)^{\frac{1}{0.9}}\right]^{0.9}\right] + E\left[0.2898{X_{10}}^{2.4}\varepsilon_{10}\varepsilon_D\right] + E\left[\sum_{i=2}^{9} Y_i\, D\right],$$

and $\mathrm{E}[YD] = 21918.224 + 18645.796 + 74305.896 = 114869.916$.

Now that all of the variables of the functional correlation have been obtained, the correlation can be computed as:

$$\rho_{Y,D} = \frac{E[YD] - \mu_Y\mu_D}{\sigma_Y\sigma_D} = \frac{114869.916 - (2166.995)(52.823)}{(443.915)(24.935)} = 0.0364$$

The value $\rho_{Y,D}$ calculated from a 100,000-trial statistical simulation is 0.0366, which indicates excellent agreement with the analytic result.

11.1.4.2 Joint PDF of Cost and Schedule

The joint PDF of cost and schedule is computed using a bivariate lognormal distribution. The bivariate lognormal distribution is defined by the moments of the cost and schedule distributions and the correlation between the two distributions Figure 11-7.

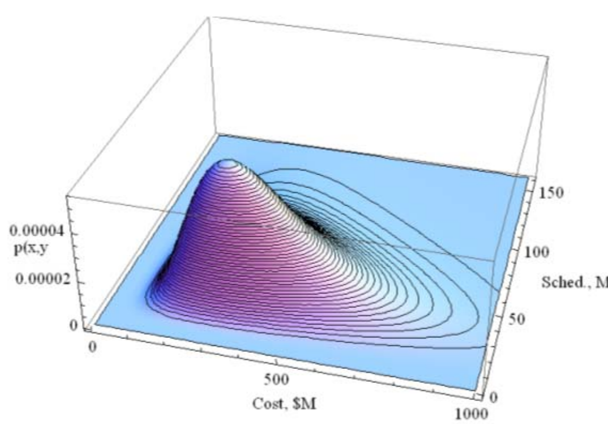

Figure 11-7 Joint PDF of Cost and Schedule

11.2 Resource-Loaded Schedule Example

NASA provided a schedule network of a rocket engine program (Table 11-7 and Figure 11-8) which will be used to demonstrate the analytic method of uncertainty analysis on a resource-loaded schedule. This example demonstrates the application of the analytic method by providing a schedule risk analysis (including a probabilistic critical path analysis), a cost risk analysis, and a joint cost and schedule risk analysis. In this section we show how we developed the cost PDF, schedule PDF, joint cost and schedule PDF and a probabilistic critical path analysis for the program.

Table 11-7 NASA Resource-Loaded Schedule Example

ID	Task	Duration	Start	Finish	Predecessor	Successor
1	**Analysis File**	840 days	10/1/2012	12/18/2015		
2	**Milestone Summary**	840 days	10/1/2012	12/18/2015		
3	Project ATP	0 days	10/1/2012	10/1/2012		11,8SS
4	PDR	0 days	4/26/2013	4/26/2013	12FF	
5	CDR	0 days	10/24/2014	10/24/2014	20FF	
6	Rocket delivery	0 days	12/18/2015	12/18/2015	32FF	9FF
7	**Project Support Costs hammock task**	840 days	10/1/2012	12/18/2015		
8	Support Start	0 days	10/1/2012	10/1/2012	3SS	
9	Support Finish	0 days	12/18/2015	12/18/2015	6FF	
10	**Preliminary Design**	150 days	10/1/2012	4/26/2013		
11	Requirements definition and documentation	100 days	10/1/2012	2/15/2013	3	12
12	Preliminary design activities	50 days	2/18/2013	4/26/2013	11	14,4FF
13	**Detailed Design**	390 days	4/29/2013	10/24/2014		
14	Initial detailed design	80 days	4/29/2013	8/16/2013	12	15,16
15	Design GN&C	160 days	8/19/2013	3/28/2014	14	20
16	Trade studies and analysis	60 days	8/19/2013	11/8/2013	14	17,18,19,35
17	Design pyrotechnics	100 days	11/11/2013	3/28/2014	16,35	20
18	Design propulsion system	160 days	11/11/2013	6/20/2014	16,35	20
19	Design structures and mechanisms	120 days	11/11/2013	4/25/2014	16,35	20
20	Finalize integrated design	90 days	6/23/2014	10/24/2014	17,18,15,19	25,5FF,23,24
21	**Development and Unit Testing**	150 days	10/27/2014	5/22/2015		
22	**Fabricate rocket Components**	120 days	10/27/2014	4/10/2015		
23	Fabricate and unit test structure (including pyros)	120 days	10/27/2014	4/10/2015	20	27
24	Fabricate and unit test engine	120 days	10/27/2014	4/10/2015	20	27,34
25	Develop and test flight software for GN&C	150 days	10/27/2014	5/22/2015	20	29,36
26	**Integration and Testing**	170 days	4/13/2015	12/4/2015		
27	Integrate rocket components	40 days	4/13/2015	6/5/2015	23,24,34	28,29
28	Test frame, fuel system and engine	35 days	6/8/2015	7/24/2015	27	30
29	Test guidance system	60 days	6/8/2015	8/28/2015	25,27,36	30
30	Final integration and testing	70 days	8/31/2015	12/4/2015	28,29	32
31	**Delivery**	10 days	12/7/2015	12/18/2015		
32	Delivery	10 days	12/7/2015	12/18/2015	30	6FF
33	**Risk Register**	400 days	11/8/2013	5/22/2015		
34	Risk 1 - TI - Additional Purchase	0 days	4/10/2015	4/10/2015	24	27
35	Risk 2 - Duration - Additional Studies Required	0 days	11/8/2013	11/8/2013	16	17,18,19
36	Risk 3 - TI and Duration - Delay from Additional SW Purchase	0 days	5/22/2015	5/22/2015	25	29

The nominal start and finish dates for the program are 10/1/2012 (defined by task 3, which is the project's ATP date), and 12/18/2015 (defined by tasks 6 and 32 which are the tasks that define the delivery date), respectively. Using the nominal dates and durations, we get a point estimate of schedule duration equal to 1173 calendar days. The milestone summaries are tasks 2 through 6; the program support "hammock tasks" are tasks 7 through 9 whose duration is defined by the ATP and delivery dates; the design, development, integration, test and delivery tasks are tasks 10 through 32; and the program risks are tasks 33 through 36. The Gantt chart for this schedule is shown in Figure 11-8.

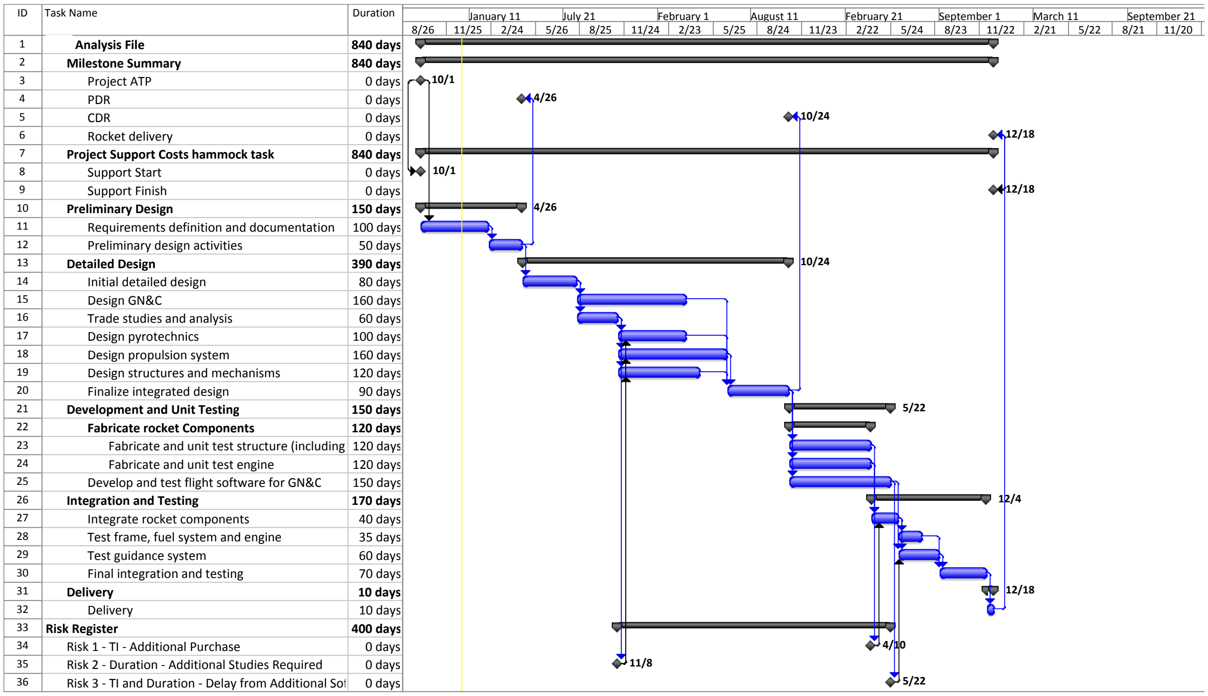

Figure 11-8 NASA Resource-Loaded Schedule Example Gantt Chart

The nominal critical path summary tasks include “Preliminary Design” (task 10), “Detailed Design” (task 13), “Development and Unit Testing” (task 21), “Integration and Testing” (task 26), “Delivery” (task 31), and “Risk Register” (schedule-related risks summarized by task 33).

11.2.1 Schedule Probability Distribution

The schedule distribution will be defined by the distributions of those tasks on the probabilistic critical paths (i.e., tasks 10 through 36). We will define the probabilistic finish dates of these tasks using Equation 11-1.

$$Finish_i = Start_i + Duration_i \text{ where } i \text{ is the task number} \quad \textbf{11-1}$$

11.2.1.1 Input Probability Distributions

The ATP date is defined as a discrete date. The remaining start and finish dates for all of the tasks are RVs because each of the task durations are defined as RVs with parameters defined in Table 11-8. Two types of PDFs are shown in Table 11-8. The first type of PDF is used to replace the nominal duration with a RV and are defined as uniform (U), triangular (T), normal (N) or lognormal (L) PDFs. The second type of PDF is an uncertainty used to multiply the nominal duration by a PDF. These are identified with the same PDF shape symbols as the first (e.g., U, T, N, L), but have a multiplication symbol next to the distribution's parameters (e.g., $U*(90{,}110)$), which are represented as percentile values.

All of the PDFs are uncorrelated with respect to each other except for tasks 23, 24 and 25 (i.e., the development duration "*DEVDUR*" tasks). These tasks are correlated with each other with $\rho = 0.75$.

Table 11-8 Duration Probability Distributions

Task ID	Task Description	PDF
7	**Project Support Costs hammock task**	
10	**Preliminary Design**	
11	Requirements definition and documentation	$T*(95{,}100{,}110)$
12	Preliminary design activities	$T*(95{,}100{,}110)$
13	**Detailed Design**	
14	Initial detailed design	$T*(90{,}100{,}120)$
15	Design GN&C	$T*(90{,}100{,}120)$
16	Trade studies and analysis	$T*(90{,}100{,}120)$
17	Design pyrotechnics	$T*(90{,}100{,}120)$
18	Design propulsion system	$T*(90{,}100{,}120)$
19	Design structures and mechanisms	$T*(90{,}100{,}120)$
20	Finalize integrated design	$T*(90{,}100{,}120)$
21	**Development and Unit Testing**	
22	**Fabricate Rocket Components**	
23	Fabricate and unit test structure (including pyros)	$U*(80{,}110)$; $\rho(DEVDUR = 0.75)$
24	Fabricate and unit test engine	$U*(80{,}110)$; $\rho(DEVDUR = 0.75)$
25	Develop and test flight software for GN&C	$L*(105{,}5)$; $\rho(DEVDUR = 0.75)$
26	**Integration and Testing**	
27	Integrate rocket components	$N*(100{,}15)$
28	Test frame, fuel system and engine	$T*(80{,}100{,}130)$
29	Test guidance system	$T*(80{,}100{,}130)$
30	Final integration and testing	$T(55{,}70{,}91)$
31	**Delivery**	
32	Delivery	$N(10{,}3)$
33	**Risk Register**	
34	Risk 1 - TI - Additional Purchase	$R(p,D)(0.30{,}0)$
35	Risk 2 - Duration - Additional Studies Required	$R(p,D)(0.15, DU(15{,}25{,}40))$
36	Risk 3 - TI and Duration - Delay from Additional Software Purchase	$R(p,D)(0.3, T(20{,}25{,}30))$

11.2.1.2 Calculating the Schedule Probability Distributions

Using the discrete start date of 10/01/2012, the predecessor/successor relationships defined in Table 11-7 and the probabilistic durations of the tasks defined in Table 11-8, we can find the PDF of the finish dates of the resource-loaded example schedule.

Only one obstacle lies in our way – the issue of whether to compute the statistics in working days or calendar days. For simplicity, we will perform computations in working days - denoting durations, start dates and finish dates with an accent (e.g., $finish'_i$) - then when specific dates are required, convert them to calendar days using the conversion factor in Equation 3-8.

An example calculation of the duration statistics follows: Since $Duration'_{11}$ is a PDF defined by $100wd * T(95{,}100{,}110)/100$, $\mu_{Duration'_{11}} = 101.67\ wd$ and $\sigma_{Duration'_{11}} = 3.12wd$ using the definitions of the mean and standard deviation of a triangular PDF from Section 16.1.1. We repeat these calculations to compute the duration statistics (in wd) for all non-summary tasks shown in Table 11-9.

The discrete risk duration calculations for tasks 34-36 rely on the technique described in Section 9. There are two risks, R_2 and R_3, with which we are currently concerned. R_2 is defined as a discrete risk, $R_2(0.15, D(15{,}25{,}40))$, with probability of occurrence of 15% and equiprobable consequences of 15, 25, and 40 wd, respectively. The possible outcomes and associated probabilities of the states of R_2 are:

$$Duration'_{R_2} = \begin{cases} 0\ wd & , p = 0.85 \\ 15\ wd & , p = 0.05 \\ 25\ wd & , p = 0.05 \\ 40\ wd & , p = 0.05 \end{cases}$$

The moments of the duration of R_2 are calculated using Equations 9-4 and 9-7.

$\mu_{Duration'_{R_2}} = p\left(\frac{D_1+D_2+D_2}{3}\right) = 0.15\left(\frac{15+25+40}{3}\right) = 4wd$, and

$$\sigma_{Duration'_{R_2}} = \sqrt{(1-p)\left(D_0 - \mu_{Duration'_{R_2}}\right)^2 + \frac{p}{3}\sum_{i=1}^{3}\left(D_i - \mu_{Duration'_{R_2}}\right)^2}$$

$$\sigma_{Duration'_{R_2}} = \sqrt{(1-0.15)(0-4)^2 + \frac{0.15}{3}[(15-4)^2 + (25-4)^2 + (40-4)^2]}$$

$$\sigma_{Duration'_{R_2}} = \sqrt{(0.85)(16) + 0.05[(11)^2 + (21)^2 + (36)^2]} = 10.32wd.$$

Task 36 (R_3) is defined as a discrete risk, $R_3(0.30, T(20{,}25{,}30))$, with probability of occurrence of 30% and a probabilistic impact, D, defined by a triangular distribution with parameters 20, 25 and 30 wd, respectively. The mean and standard deviation of the impact's triangular PDF, $D = T(20,25,30)$, are:

$\mu_D = \frac{\sum_{i=1}^{3} D_i}{3} = \frac{20+25+30}{3} = 25wd$, and

$\sigma_D = \sqrt{\frac{\sum_{i=1}^{3}(D_i - \mu_D)^2}{12}} = \sqrt{\frac{(20-25)^2+(25-25)^2+(30-25)^2}{12}} = 2.04wd.$

The moments of the duration of R_3 are calculated using Equations 9-4 and 9-7

$\mu_{Duration'_{R3}} = p\mu_D = 0.30(25) = 7.50wd$, and

$$\sigma_{Duration'_{R3}} = \sqrt{(1-p)\left(D_0 - \mu_{Duration'_{R3}}\right)^2 + p\left[\sigma_D{}^2 + \sum_{i=1}^{3}\left(D_i - \mu_{Duration'_{R3}}\right)^2\right]}$$

$\sigma_{Duration'_{R3}} =$
$\sqrt{(1-0.30)(0-7.5)^2 + 0.3[(2.04)^2 + (20-7.5)^2 + (25-7.5)^2 + (30-7.5)^2]} =$
$11.51wd.$

Table 11-9 Duration Probability Distributions in Workdays

Task ID	Duration, wd	PDF, ε_i	μ_{ε_i}	σ_{ε_i}
11	100	$T*(95,100,110)$	101.67	3.12
12	50	$T*(95,100,110)$	50.83	1.56
14	80	$T*(90,100,120)$	82.67	4.99
15	160	$T*(90,100,120)$	165.33	9.98
16	60	$T*(90,100,120)$	62.00	3.74
17	100	$T*(90,100,120)$	103.33	6.24
18	160	$T*(90,100,120)$	165.33	9.98
19	120	$T*(90,100,120)$	124.00	7.48
20	90	$T*(90,100,120)$	93.00	5.61
23	120	$U*(80,110)$	114.00	10.39
24	120	$U*(80,110)$	114.00	10.39
25	150	$L*(105,5)$	157.5	7.50
27	40	$N*(100,15)$	40	6.00
28	35	$T*(80,100,130)$	36.17	3.60
29	60	$T*(80,100,130)$	62.00	6.16
30	70	$T(55,70,91)$	72.00	7.38
32	10	$N(10,3)$	10.00	3.00
34	0	$R(p,D)(0.30,0)$	0.00	0.00
35	0	$R(p,D)(0.15,DU(15,25,40))$	4.00	10.32
36	0	$R(p,D)(0.3,T(20,25,30))$	7.50	11.51

In the next series of calculations, we compute the means and standard deviations of start dates (in wd) and finish dates (in wd) for these tasks.

11.2.1.3 Preliminary Design

The "Preliminary Design" summary task (task 10) consists to two lowest-level tasks (tasks 11 and 12) that are arranged serially. The computations for the task durations, start dates and finish dates are:

$Start_{10} \equiv Start_{11} \equiv Start_3$ = ATP date of 10/01/12

$Finish'_{10} \equiv Finish'_{12}$ by definition because task 10 is a summary task

$Duration'_{10} = Finish'_{10} - Start_{10} = Duration'_{11} + Duration'_{12}$, because tasks 11 and 12 are serial tasks

$Finish'_{11} = Start_{11} + Duration'_{11}$, in wd

From Table 11-9 we have:

$\mu_{Duration'_{11}} = 101.67wd$ and $\sigma_{Duration'_{11}} = 3.12wd$
$\mu_{Duration'_{12}} = 50.83wd$ and $\sigma_{Duration'_{12}} = 1.56wd.$

So, $Finish'_{12} = Start_{12} + Duration'_{12} = Finish_{11} + Duration'_{12} = Start_3 + Duration'_{11} + Duration'_{12}$, in wd

So $Duration'_{10} = Duration'_{11} + Duration'_{12}$

Therefore,

$$\mu_{Duration'_{10}} = \mu_{Duration'_{11}} + \mu_{Duration'_{12}} = 152.50wd, \text{ and}$$
$$\sigma_{Duration'_{12}} = \sqrt{\sigma_{Duration'_{11}}{}^2 + \sigma_{Duration'_{12}}{}^2} = 3.49wd.$$

Using these calculations, we get the results in Table 11-10.

Table 11-10 Workday Results for Preliminary Design

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
10	152.50	3.49	10/01/12	0	03/02/13	3.49
11	101.67	3.12	10/01/12	0	01/10/13	3.12
12	50.83	1.56	01/10/13	3.12	03/02/13	3.49

11.2.1.4 Detailed Design

The "Detailed Design" summary task (task 13) consists of seven lowest-level tasks (tasks 14 through 20) arranged in a tree structure. The nominal durations of tasks 14 through 20 have the same multiplicative triangular PDF (defined by $Duration' * T(90{,}100{,}120)$), with mean of 1.033 and a standard deviation of 0.062.

Task 14 has one predecessor, task 12, so its start and finish dates are defined as:

$Start'_{14} = Finish'_{12}$, and $Finish'_{14} = Finish'_{12} + Duration'_{14}$

From Table 11-9:

$\mu_{Duration'_{14}} = 80 * 1.033 = 82.67wd$, and
$\sigma_{Duration'_{14}} = 80 * 0.062 = 4.99wd.$

Table 11-11 Workday Results for Detailed Design Task 14

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
14	82.67	4.99	03/02/13	3.49	05/24/13	6.09

Task 15 has a single predecessor, task 14, and we compute its start and finish dates as:

$Start'_{15} = Finish'_{14}$, and $Finish'_{15} = Start'_{15} + Duration'_{15}$

From Table 11-9:

$\mu_{Duration'_{15}} = 160 * 1.033 = 165.33wd$, and
$\sigma_{Duration'_{15}} = 160 * 0.062 = 9.98wd.$

Task 16 also has a single predecessor (task 14), and its start and finish dates are:

$Start'_{16} = Finish'_{14}$, and $Finish'_{16} = Finish'_{14} + Duration'_{16}$

From Table 11-9:

$\mu_{Duration'_{16}} = 60 * 1.033 = 62.00wd$, and
$\sigma_{Duration'_{16}} = 60 * 0.062 = 3.74wd.$

Tasks 17 through 19 share risk R_2 as a common predecessor, and R_2's predecessor is task 16. We must first compute the moments of R_2 in order to calculate the start dates, durations and end dates of tasks 17 through 19.

So, $Start'_{R_2} = Finish'_{16}$ and $Finish'_{R2} = Finish'_{16} + Duration'_{R_2}$ and

From Table 11-9:

$\mu_{Duration'_{R_2}} = 4wd$ and $\sigma_{Duration'_{R_2}} = 10.32wd.$

Since $Start'_{17} = Start'_{18} = Start'_{19} = Finish'_{R_2}$, and

$Finish'_{17} = Finish'_{R_2} + Duration'_{17}$, $Finish'_{18} = Finish'_{R_2} + Duration'_{18}$, and

$Finish'_{19} = Finish'_{R_2} + Duration'_{18}$, we need to compute the moments of the durations of tasks 17 through 19 to compute their finish dates.

From Table 11-9:

$\mu_{Duration'_{17}} = 103.33wd$, $\sigma_{Duration'_{17}} = 6.24wd$,
$\mu_{Duration'_{18}} = 165.33wd$, $\sigma_{Duration'_{18}} = 9.98wd$,
$\mu_{Duration'_{19}} = 124.00wd$, and $\sigma_{Duration'_{19}} = 7.48wd$.

The statistics for the durations, start dates and end dates for tasks 15 through 19 (including task 36) are shown in Table 11-12.

Table 11-12 Workday Results for Detailed Design Tasks 15 – 19 and 35

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
15	165.33	9.98	05/24/13	6.09	11/05/13	11.69
16	62.00	3.74	05/24/13	6.09	07/25/13	7.14
35 (R_2)	4.00	10.32	07/25/13	7.14	07/29/13	12.55
17	103.33	6.24	07/29/13	12.55	11/09/13	14.02
18	165.33	9.98	07/29/13	12.55	01/10/14	16.03
19	124.00	7.48	07/29/13	12.55	11/30/13	14.61

Task 20 has four predecessor tasks, so its start date is defined by the maximum finish date of its predecessors (i.e., tasks 15, 17, 18 and 19). This is expressed as:

$Start'_{20} = Max(Finish'_{15}, Finish'_{17}, Finish'_{18}, Finish'_{19})$

Nearly all of the duration PDFs used in this example schedule are right skewed, so a lognormal distribution is assumed for all of the start and finish date PDFs. Since the distributions of the finish dates of these tasks approximate lognormal distributions, the equations for the moments of the maximum of lognormal distributions (Equations 10-8 through 10-10) are used to find the finish date statistics for tasks 15, 17, 18 and 19 and thus the start date statistics for task 20. The latest, or maximum, finish date of the four tasks can be calculated in pairs, so three comparisons will be made, and the following three intermediate distributions will be formed: $A = max(18,19)$, $B = max(17, A)$, and $C = max(15, B)$.

We can calculate the mean of the maximum of two lognormal distributions using Equation 10-8, which is:

$$[X] = \mu_1 \Phi\left[\frac{(P_1 - P_2) + (Q_1{}^2 - \rho Q_1 Q_2)}{\theta}\right] + \mu_2 \Phi\left[\frac{(P_2 - P_1) + (Q_2{}^2 - \rho Q_1 Q_2)}{\theta}\right],$$

and from Equation 10-9, which is:

$$E[X^2] = (\sigma_1^2 + \mu_1^2)\Phi\left(\frac{P_1 - P_2}{\theta}\right) + (\sigma_2^2 + \mu_2^2)\Phi\left(\frac{P_2 - P_1}{\theta}\right),$$

where $= \sqrt{Q_1^2 + Q_2^2 - 2\rho Q_1 Q_2}$, and $\rho = \frac{1}{Q_1 Q_2} ln\left[1 + \rho_{1,2}\left(\sqrt{[e^{Q_1^2} - 1][e^{Q_2^2} - 1]}\right)\right]$

These computations require knowledge of the statistics of the finish dates of pairs of tasks: $\mu_1, \mu_2, \sigma_1, \sigma_2, P_1, P_2, Q_1, Q_2, \theta, \rho_{1,2}$, and ρ. Table 11-13 provides the statistics used in the calculation of the maximum finish dates of tasks 15, 17, 18 and 19.

The finish dates of tasks 15 through 19 are correlated due to common predecessor-successor relationships. Using Equation 8-8, we can determine the pairwise correlation between these tasks or the maximums of pairs of tasks.

Table 11-13 Statistics for Maximum Finish Dates of Tasks 15, 17, 18 and 19

Statistic	A=max(18,19)	B=max(17,A)	C=max(15,B)
μ_1	01/10/14	11/09/13	11/05/13
μ_2	11/30/13	01/10/14	01/10/14
σ_1	16.03	14.02	11.69
σ_2	14.61	16.03	16.03
P_1	10.6370	10.6356	10.6355
P_2	10.6361	10.6370	10.6370
Q_1	0.000385	0.000337	0.000281
Q_2	0.000351	0.000385	0.000385
$\rho_{1,2}$	0.67236	0.70115	0.14383
ρ	0.67236	0.70115	0.14383
$\mu_{\max}$	01/10/14	01/10/14	01/10/14
$\sigma_{\max}$	16.03	16.03	16.03

*Note due to the small values of Q_{i}, that $\rho_{1,2}$ and ρ are identical

Tasks 18 and 19 share a common predecessor, the risk task (task 35, or R_2), so their correlation is:

$$\rho_{18,19} = \frac{\sigma_{Finish'_{R_2}}{}^2}{\sigma_{Finish'_{18}} \sigma_{Finish'_{19}}} = \frac{(12.55)^2}{(16.03)(14.61)} = 0.67236$$

Task 17 shares R_2 as a common predecessor with the maximum of task A (the maximum of tasks 18 and 19), so the correlation between task 17 and task A is:

$$\rho_{17,A} = \frac{{\sigma_{Finish'_{R_2}}}^2}{\sigma_{Finish'_{17}}\sigma_{Finish'_{A}}} = \frac{(12.55)^2}{(14.02)(16.03)} = 0.70104$$

Finally, task 15 shares task 14 as a common predecessor with task B (the maximum of tasks 17 and A), so task 15's correlation to task B is:

$$\rho_{15,B} = \frac{{\sigma_{Finish'_{14}}}^2}{\sigma_{Finish'_{15}}{\sigma_{Finish'_{B}}}^2} = \frac{(6.09)^2}{(11.69)(16.03)} = 0.19766$$

Task 20's predecessor is task C, so its finish date is defined as:

$$Finish'_{20} = Finish'_{C} + Duration'_{20}$$

From Table 11-9:

$\mu_{Duration'_{14}} = 90 * 1.033 = 93.00wd$, and
$\sigma_{Duration'_{20}} = 90 * 0.062 = 5.61wd.$

The start date, finish date and duration results for task 20 are shown in Table 11-14.

Table 11-14 Workday Results for Detailed Design Task 20

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
20	93.00	5.61	01/10/14	16.03	04/13/14	16.99

11.2.1.5 Development and Unit Testing

The "Development and Unit Testing" summary task (task 21) consists of a summary task (task 22) and three lowest-level tasks (tasks 23 through 25) that are arranged in a parallel structure. Each of the tasks has a common predecessor, task 20 and the durations of tasks 23, 24 and 25 are correlated to each other with $\rho = 0.75$. From Table 11-9:

$\mu_{Duration'_{23}} = 114.00wd$, $\sigma_{Duration'_{23}} = 10.39wd$,
$\mu_{Duration'_{24}} = 114.00wd$, $\sigma_{Duration'_{24}} = 10.39wd$,
$\mu_{Duration'_{25}} = 157.50wd$, and $\sigma_{Duration'_{25}} = 7.50wd$.

Table 11-15 shows the duration, start and finish date statistics for the lowest-level tasks for "Development and Unit Testing".

Table 11-15 Workday Results for Development and Unit Testing Tasks

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
23	114.00	10.39	04/13/14	16.98	08/05/14	19.91
24	114.00	10.39	04/13/14	16.98	08/05/14	19.91
25	157.5	7.50	04/13/14	16.98	09/18/14	18.57

The fact that the durations of these tasks are correlated does not matter at this particular point since they are not serially arranged and merge to form a predecessor in a different summary task. This will become important when computing the maximum finish dates of these tasks and their respective CIs.

11.2.1.6 Integration and Testing

The "Integration and Testing" summary task (task 26) consists of four lowest-level tasks (27 through 30) arranged in a tree structure.

Task 27 has three predecessors, tasks 23, 24 and 34, and the last is risk R_1. Since R_1 has zero duration, task 27 actually has only two predecessors, tasks 23 and 24. This means its start date is defined as maximum of the finish of tasks 23 and 24. Both tasks 23 and 24 have the same finish statistics but their durations are correlated with $\rho = 0.75$.

$Start'_{27} = Max(Finish'_{23}, Finish'_{24})$

$Finish'_{27} = Start'_{27} + Duration'_{27}$

From Table 11-9, $\mu_{Duration'_{27}} = 40.00wd$ and $\sigma_{Duration'_{27}} = 6.00wd$.

Since the maximum of $Finish'_{23}$ and $Finish'_{24}$ depends on the correlation between the durations of tasks 23 and 24 as well as the functional correlation due to their common predecessor (task 2), we will use Equation 8-8 to determine $\rho_{F'_{23},F'_{24}}$then we can calculate the maximum finish date statistics using Equations 10-8 through 10-10.

$$\rho_{F'_{23},F_{24}} = \frac{{\sigma_{F'_{20}}}^2 + \rho_{D'_{23},D'_{24}}\sigma_{D'_{23}}\sigma_{D'_{24}}}{\sigma_{F'_{23}}\sigma_{F'_{24}}} = \frac{(16.99)^2 + (0.75)(10.39)(10.39)}{(19.91)(19.91)} = 0.9319$$

The maximum finish date statistics are:

$\mu_{(Finish'_{23},Finish'_{24})} = 08/08/14$, and $\sigma_{(Finish'_{23},Finish'_{24})} = 19.70wd$.

Task 27's statistics are provided in Table 11-16.

Table 11-16 Workday Results for Integration and Testing Task 27

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
27	40	6.00	08/08/14	19.70	09/17/14	20.59

Task 28 has a single predecessor (task 27), so $Finish'_{27} = Start'_{28}$. From Table 11-9:

$\mu_{Duration'_{28}} = 36.17wd$ and $\sigma_{Duration'_{28}} = 3.60wd$.

Task 28's statistics are provided in Table 11-17.

Table 11-17 Workday Results for Integration and Testing Task 28

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
28	36.17	3.60	09/17/14	20.59	10/23/14	20.90

The "Development and Unit Testing" task (task 29), has two predecessors, tasks 27 and 36 (the latter is risk R_3). Tasks 27 and 36 branch from task 20 with multiple intermediate tasks, but since they share task 20 as a common predecessor, their finish dates will be functionally correlated. Before we can compute task 29's start date, we must compute the finish date statistics for task 36. From Table 11-9:

$\mu_{Duration'_{R3}} = 7.50wd$ and $\sigma_{Duration'_{R3}} = 11.51wd$.

The predecessor-successor and start-finish relationships ($Start'_{R3} = Finish'_{25}$ and $Finish'_{R3} = Start'_{R3} + Duration'_{R3}$) allow us to compute the schedule statistics for task 36 (R_3) in Table 11-18.

Table 11-18 Workday Results for Risk R3 (Task 36)

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
36	7.50	11.51	09/18/14	18.57	09/25/14	21.85

Since tasks 27 and 36 share a common predecessor (task 20), they are functionally correlated, so we will use the now familiar Equation 8-8 to determine $\rho_{F'_{27},F'_{36}}$. Since $\rho_{D_{27},D_{36}} = 0.75$, we have:

$$\rho_{F'_{27},F'_{36}} = \frac{\sigma_{F'_{20}}{}^2 + \rho_{D'_{27},D'_{36}}\sigma_{D'_{27}}\sigma_{D'_{36}}}{\sigma_{F'_{27}}\sigma_{F'_{36}}} = \frac{(16.99)^2 + (0.75)(6.00)(11.51)}{(20.59)(21.85)} = 0.7565$$

The maximum finish date statistics using Equations 10-8 through 10-10 are

$\mu_{(Finish'_{27},Finish'_{36})} = 09/28/14$, and $\sigma_{(Finish'_{23},Finish'_{24})} = 20.78wd$.

From Table 11-9:

$\mu_{Duration'_{29}} = 62.00wd$ and $\sigma_{Duration'_{29}} = 6.16wd$.

Task 29's statistics are provided in Table 11-19.

Table 11-19 Workday Results for Integration and Testing Task 29

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
29	62.00	6.16	09/28/14	20.78	11/29/14	21.68

The last "Integration and Testing" task is task 30. It has two predecessors, tasks 28 and 29. Since tasks 28 and 29 share task 27 as a common predecessor, they will be functionally correlated, and we will use Equation 8-8 to calculate it. We will assume $\rho_{D_{28},D_{29}} = 0$.

$$\rho_{D'_{28},D'_{29}} = \frac{{\sigma_{F'_{27}}}^2}{\sigma_{F'_{28}}\sigma_{F'_{28}}} = \frac{(20.59)^2}{(20.90)(21.68)} = 0.9356$$

Equations 10-8 through 10-10 provide the following results

$\mu_{(Finish'_{28},Finish'_{29})} = 11/30/14$, and $\sigma_{(Finish'_{28},Finish'_{29})} = 21.68wd$.

From Table 11-9:

$\mu_{Duration'_{30}} = 72.00wd$ and $\sigma_{Duration'_{30}} = 7.38wd$.

Table 11-20 summarizes the duration, start and finish date statistics for the lowest-level tasks for "Integration and Testing".

Table 11-20 Workday Results for Integration and Testing Tasks

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
27	40	6.00	08/08/14	19.70	09/17/14	20.59
28	36.17	3.60	09/17/14	20.59	10/23/14	20.90
29	62.00	6.16	09/28/14	20.78	11/29/14	21.68
30	72.00	7.38	11/30/14	21.68	02/10/15	22.90

11.2.1.7 Delivery

The "Delivery" summary task (task 31) consists of a single lowest-level task (task 32). Task 32 has a single predecessor (task 30). From Table 11-9:

$\mu_{Duration'_{32}} = 10.00wd$, and $\sigma_{Duration'_{32}} = 3.00wd$.

The statistics for task 32 are shown in Table 11-21.

Table 11-21 Workday Results for Delivery Task 32

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$
32	10.00	3.00	02/10/15	22.90	02/20/15	23.10

11.2.1.8 Criticality Index

As described in Section 3.3.3, the CI is the probability that a particular task is on the critical path. Since tasks 30 and 32 are serial and always define the finish date, they are always on the critical path so their CIs are 100%. Tasks 11, 12 and 14 are serial tasks and are by definition on the critical path so their CIs are 100% as well. The tasks succeeding task 14 create a branch in the schedule network, so we must evaluate their CI up to the point of the start of task 20. These branches are:

1) Task 15
2) Tasks 16, 35 and 17
3) Tasks 16, 35 and 18
4) Tasks 16, 35 and 19

The expression for the duration between task 15 and task 19 is:

$$D_{[15,19]} = \max(D_{15}, D_{16} + D_{35} + D_{17}, D_{16} + D_{35} + D_{18}, D_{16} + D_{35} + D_{19})$$

The CIs of these tasks, using Equation 3-9, are:

$$CI_{15} = P(F'_{15} > F'_{max(17,18,19)})$$
$$CI_{16} = 1 - CI_{15}$$
$$CI_{35} = 1 - CI_{15}$$
$$CI_{17} = P(F'_{17} > F'_{max(18,19)})$$
$$CI_{18} = P(F'_{18} > F'_{max(17,19)})$$
$$CI_{19} = P(F'_{19} > F'_{max(17,18)})$$

From Section 3.3.3, we can calculate CI_{15} using the moments of the difference between the PDFs of $F'_{\max(17,18,19)}$ and F'_{15} then finding the integral of the PDF of the difference from $-\infty$ to 0.

$$CI_{15} = P\left(F'_{15} > F'_{\max(17,18,19)}\right) = P(F'_{\max(17,18,19)} - F'_{15} < 0)$$

$\mu_{F'_{\max(17,18,19)}} = 01/10/14$, and $\sigma_{F'_{\max(17,18,19)}} = 16.03wd$ from Table 11-13.
$\mu_{F'_{15}} = 11/05/13$, and $\sigma_{F'_{15}} = 11.69\ wd$ from Table 11-12
$\rho = 0.14383$ from Table 11-13.

The moments of the difference of the PDFs are:

$\delta\mu = \mu_{F'_{\max(17,18,19)}} - \mu_{F'_{15}} = 66wd$, and

$$\delta\sigma = \sqrt{{\sigma_{F'_{\max(17,18,19)}}}^2 + {\sigma_{F'_{15}}}^2 - 2\rho\sigma_{F'_{\max(17,18,19)}}\sigma_{F'_{15}}} = 17.875wd.$$

Since $\delta\mu$ is positive and $\sigma_{F'_{15}} < \sigma_{F'_{\max(17,18,19)}}$, we expect the difference distribution to be right skewed. Using the knowledge that $\delta\mu > 3\delta\sigma$ we can expect all but a negligible amount of area of the distribution lies to the right of the origin, so $CI_{15} = 0$. Since this is the case, $CI_{16} = 1$, and $CI_{35} = 1$ since it is a direct successor to task 16.

$CI_{17} = P\left(F'_{17} > F'_{\max(18,19)}\right) = P(F'_{\max(18,19)} - F'_{17} < 0)$

> $\mu_{F'_{\max(18,19)}} = 01/10/14$, and $\sigma_{F'_{\max(18,19)}} = 16.03wd$ from Table 11-13.
> $\mu_{F'_{17}} = 11/09/13$, and $\sigma_{F'_{17}} = 14.02wd$ from Table 11-12
> $\rho = 0.70115$ from Table 11-13.

The moments of the difference of the PDFs are:

$\delta\mu = \mu_{F'_{\max(18,19)}} - \mu_{F'_{15}} = 62\ wd.$, and

$$\delta\sigma = \sqrt{\sigma_{F'_{\max(18,19)}}{}^2 + \sigma_{F'_{15}}{}^2 - 2\rho\sigma_{F'_{\max(18,19)}}\sigma_{F'_{15}}} = 11.763\ wd.$$

Again, $\delta\mu$ is positive, and $\sigma_{F'_{17}} < \sigma_{F'_{\max(17,18,19)}}$, so we expect the difference distribution to be right skewed. $\delta\mu > 3\delta\sigma$, in this case, so we can again expect $CI_{17} = 0$.

$CI_{18} = P\left(F'_{18} > F'_{\max(17,19)}\right) = P(F'_{\max(17,19)} - F'_{18} < 0)$

To find CI_{18} we require values for the following parameters: $\mu_{F'_{\max(17,19)}}$, $\sigma_{F'_{\max(18,19)}}$, $\mu_{F'_{18}}$, $\sigma_{F'_{18}}$, and ρ (which is the correlation between $F'_{\max(17,19)}$ and F'_{18}).

We again use Equation 10-8 and Equation 10-9 to calculate the mean of $F'_{\max(17,19)}$ which result in:

> $\mu_{F'_{\max(17,19)}} = 11/30/13$, and $\sigma_{F'_{\max(18,19)}} = 14.56\ wd$
> $\mu_{F'_{18}} = 01/10/14$, and $\sigma_{F'_{18}} = 16.03wd$ from Table 11-12.

The correlation coefficient is calculated using the knowledge that these distributions rely on a common finish date for task 20 whose standard deviation is:

> $\sigma_{F'_{18}} = 16.99\ wd$ from Table 11-12

So, $\rho_{17,19} = \frac{\sigma_{Finish'_{20}}{}^2}{\sigma_{F'_{\max(17,19)}}\sigma_{F'_{18}}} = \frac{(16.99)^2}{(14.56)(16.03)} = 0.70115$

The moments of the difference of the PDFs are:

$\delta\mu = \mu_{F'_{\max(17,19)}} - \mu_{F'_{18}} = -41.27wd$, and

$$\delta\sigma = \sqrt{\sigma_{F'_{\max(17,19)}}{}^2 + \sigma_{F'_{18}}{}^2 - 2\rho\sigma_{F'_{\max(17,19)}}\sigma_{F'_{18}}} = 12.411\ wd.$$

The area of this distribution is all less than zero, so $CI_{18} = 1.0$.

Task 20 is a serial task and has a CI of 100%, but it has a complex set of branches succeeding it. The equivalent duration of the tasks between tasks 20 and 30 is the difference between the start date of task 30 and the finish date of task 20. This duration represents the maximum duration of tasks 23 through 29, $D_{[23,29]}$, which is equal to:

$$D_{[23,29]} = max\{max[max(D'_{23}, D'_{24}) + D'_{27}, D'_{25} + D'_{36}] + D'_{29}, max(D'_{23}, D'_{24}) + D'_{27} + D'_{28}\}$$

Tasks 28 and 29 define the start of task 30, so

$CI_{28} = P(F'_{28} > F'_{29})$, and
$CI_{29} = 1 - CI_{28}$.

$CI_{28} = P(F'_{28} > F'_{29}) = P(F'_{28} - F'_{27} < 0)$, which results in:

$\mu_{F'_{28}} = 10/23/14$, and $\sigma_{F'_{28}} = 20.89wd$ from Table 11-20
$\mu_{F'_{29}} = 11/29/14$, and $\sigma_{F'_{29}} = 21.68wd$ from Table 11-20

Since tasks 28 and 29 share task 20 as a common predecessor, the correlation between their finish dates is defined as:

$$\rho_{28,29} = \frac{\sigma_{Finish'_{20}}{}^2}{\sigma_{F'_{28}}\sigma_{F'_{29}}} = \frac{(16.99)^2}{(20.69)(21.68)} = 0.63640$$

The moments of the difference between the two PDFs are:

$\delta\mu = \mu_{F'_{29}} - \mu_{F'_{28}} = 36.6wd$, and

$$\delta\sigma = \sqrt{\sigma_{F'_{29}}{}^2 + \sigma_{F'_{28}}{}^2 - 2\rho\sigma_{F'_{29}}\sigma_{F'_{28}}} = 18.164wd.$$

Since $\rho_{28,29}$ is not large enough to model the difference between these PDFs as a normal distribution, we will treat it as a lognormal distribution. The lognormal parameters P and Q for the difference are $P = 3.4915$ and $Q = 0.4687$. Substituting P and Q into the standard normal distribution and evaluating the integral of the difference of the PDFs from $-\infty$ to 0 we get zero, so $CI_{28} = 0$. It becomes clear that task 29 is on the critical path with $CI_{29} = 1$ and the expression for the duration from task 23 to task 29 reduces to:

$$D_{[23,29]} = max[max(D_{23}, D_{24}) + D_{27}, D_{25} + D_{36}] + D_{29}$$

This expression shows we must calculate CI_{36} , a discrete risk. R_3 is defined as $R(p, D) = (0.3, T(20,25,30))$wd, meaning there is a 30% probability that there will be an additional duration defined by $T(20,25,30)$wd.

The duration statistics for task 36 are:

$$\mu_{D'_{36}} = \begin{cases} 25.00 & , if\ R_3, p = 0.3 \\ 0.00 & , if\ \overline{R_3}, 1 - p = 0.7 \end{cases}, \text{ and } \sigma_{D'_{36}} = \begin{cases} 2.04 & , if\ R_3, p = 0.3 \\ 0.00 & , if\ \overline{R_3}, 1 - p = 0.7 \end{cases}$$

The finish date statistics for task 36 are:

$\mu_{F'_{36}} = \begin{cases} 10/13/14 & , if\ R_3\ , p = 0.3 \\ 09/18/14 & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$, and $\sigma_{F'_{36}} = \begin{cases} 18.67 & , if\ R_3\ , p = 0.3 \\ 18.56 & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$

$CI_{36} = P(F'_{36} > F'_{27}) = P(F'_{27} - F'_{36} < 0)$

$\mu_{F'_{27}} = 09/17/14$, and $\sigma_{F'_{27}} = 20.59wd$ from Table 11-20

Tasks 27 and 36 share task 20 as a common predecessor, however

$Finish'_{27} = Finish'_{20} + Max(D'_{23}, D'_{24}) + D'_{27}$,

$Finish'_{R3} = Finish'_{20} + D'_{25} + D'_{R3}$, and

$\rho_{23,24} = \rho_{23,25} = \rho_{24,25} = 0.75$,

so there is additional correlation for which we must account when computing $\rho_{27,36}$.

$D'_A = Max(D'_{23}, D'_{24}) + D'_{27}$, and $D'_B = D'_{25} + D'_{R3}$

$\rho_{27,36} = \frac{\sigma_{Finish'_{20}}{}^2 + \rho_{D'_A,D'_B}\sigma_{D'_A}\sigma_{D'_B}}{\sigma_{F'_{27}}\sigma_{F'_{36}}}$, which will is calculated separately for each possible outcome. We will assume $\rho_{D'_A,D'_B} = 0.75$ and compute the standard deviations, $\sigma_{D'_A}$ and $\sigma_{D'_B}$.

$\sigma_{Max(D'_{23},D'_{24})} = 10.23wd$, using Equation 10-9.

$$\sigma_{D'_A} = \sqrt{\left[\sigma_{Max(D'_{23},D'_{24})}\right]^2 + \left[\sigma_{D'_{27}}\right]^2} = \sqrt{[10.23]^2 + [6.00]^2} = 11.86wd,$$

$$\sigma_{D'_B} = \sqrt{\left[\sigma_{D'_{25}}\right]^2 + \left[\sigma_{D'_{R3}}\right]^2} = \begin{cases} \sqrt{[7.50]^2 + [2.04]^2} = 7.77wd & , if\ R_3\ , p = 0.3 \\ \sqrt{[7.50]^2 + [0]^2} = 7.50wd & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$$

$$\rho = \frac{\sigma_{Finish'_{20}}{}^2 + \rho_{D'_A,D'_B}\sigma_{D'_A}\sigma_{D'_B}}{\sigma_{F'_{27}}\sigma_{F'_{36}}} = \begin{cases} 0.90510 & , if\ R_3\ , p = 0.3 \\ 0.90507 & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$$

The moments of the difference of the PDFs are

$$\delta\mu = \mu_{F'_{27}} - \mu_{F'_{36}} = \begin{cases} -25.57wd & , if\ R_3\ , p = 0.3 \\ -0.57wd & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$$

$$\delta\sigma = \sqrt{\sigma_{F'_{27}}{}^2 + \sigma_{F'_{36}}{}^2 - 2\rho\sigma_{F'_{27}}\sigma_{F'_{36}}} = \begin{cases} 8.752wd & , if\ R_3\ , p = 0.3 \\ 8.753wd & , if\ \overline{R_3}\ , 1-p = 0.7 \end{cases}$$

In the case where R_3 occurs, we will assume the PDF of the difference is approximately normal since the two distributions are so highly correlated ($\rho = 0.90510$). Given this, the integral of the PDF of the difference from $-\infty$ to 0 is 0.9983, which is almost unity. In the case where R_3 does not occur, we will again assume a normal distribution for the PDF of the difference. The resulting integral of the PDF of the difference from $-\infty$ to 0 is 0.5259. Combining these two CIs, we get a 30% probability that CI_{36} is 0.9983 and a 70% probability that CI_{36} is 0.5259. These probabilities result in

$CI_{36} = 0.3(\,0.99830) + 0.7(0.5259) = 0.6676$, so $CI_{27} = 1 - 0.6676 = 0.3324$

The relationship for the duration $D_{[23,29]}$ can be rewritten as

$$D_{[23,29]} = \begin{Bmatrix} max(max(D_{23}, D_{24}) + D_{27}, D_{25} + D_{36}) & , p = 0.3324 \\ D_{25} + D_{36} & , 1 - p = 0.6676 \end{Bmatrix} + D_{29}$$

Since task 25 belongs to the same path as (and is a single predecessor to) task 36, then $CI_{25} = 0.6676.$

The remaining two tasks, tasks 23 and 24, have identical distributions as shown in Table 11-15, so they have an equal probability of being on the critical path. Given this we can multiply the CI of their path (CI=0.3324) by 0.5 to equally divide their probabilities of being on the critical path.

$$CI_{23} = CI_{24} = (0.5)(0.3324) = 0.1662$$

11.2.1.9 Schedule Risk Summary

Table 11-22 summarizes the duration statistics (as well as the start and finish dates in workdays) and the CIs calculated in the previous section. The durations, start and finish dates are converted to calendar dates in Table 11-23 to display the actual duration statistics in days as well as the calendar days representing the statistics of the start and finish dates of the tasks.

Table 11-22 Workday Results for Schedule Risk Analysis

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$	CI
10	152.50	3.49	10/01/12	0.00	03/02/13	3.49	100%
11	101.67	3.12	10/01/12	0.00	01/10/13	3.12	100%
12	50.83	1.56	01/10/13	3.12	03/02/13	3.49	100%
14	82.67	4.99	03/02/13	3.49	05/24/13	6.09	100%
15	165.33	9.98	05/24/13	6.09	11/05/13	11.69	0%
16	62.00	3.74	05/24/13	6.09	07/25/13	7.14	100%
17	103.33	6.24	07/29/13	12.55	11/09/13	14.02	0%
18	165.33	9.98	07/29/13	12.55	01/10/14	16.03	100%
19	124.00	7.48	07/29/13	12.55	11/30/13	14.61	0%

Task ID	$\mu_{Duration'}$	$\sigma_{Duration'}$	$\mu_{Start'}$	$\sigma_{Start'}$	$\mu_{Finish'}$	$\sigma_{Finish'}$	CI
20	93.00	5.61	01/10/14	16.03	04/13/14	16.99	100%
23	114.00	10.39	04/13/14	16.98	08/05/14	19.91	16.62%
24	114.00	10.39	04/13/14	16.98	08/05/14	19.91	16.62%
25	157.50	7.50	04/13/14	16.98	09/18/14	18.57	66.76%
27	40.00	6.00	08/08/14	19.70	09/17/14	20.59	33.24%
28	36.17	3.60	09/17/14	20.59	10/23/14	20.90	0%
29	62.00	6.16	09/28/14	20.78	11/29/14	21.68	100%
30	72.00	7.38	11/30/14	21.68	02/10/15	22.90	100%
32	10.00	3.00	02/10/15	22.90	02/20/15	23.10	100%
34	0.00	0.00	08/05/14	19.91	08/05/14	19.91	0%
35	4.00	10.32	07/25/13	7.14	07/29/13	12.55	100%
36	7.50	11.51	09/18/14	18.57	09/25/14	21.85	66.76%

Table 11-23 Calendar Day Results for Schedule Risk Analysis

Task ID	$\mu_{Duration}$	$\sigma_{Duration}$	μ_{Start}	σ_{Start}	μ_{Finish}	σ_{Finish}	CI
10	213.50	4.88	10/01/12	0.00	05/02/13	4.88	100%
11	142.33	4.37	10/01/12	0	02/20/13	4.37	100%
12	71.17	2.18	02/20/13	4.37	05/02/13	4.88	100%
14	115.73	6.98	05/02/13	4.88	08/26/13	8.52	100%
15	231.47	13.97	08/26/13	8.52	04/14/14	16.36	0%
16	86.80	5.24	08/26/13	8.52	11/21/13	10.00	100%
17	144.67	8.73	11/26/13	17.57	04/20/14	19.62	0%
18	231.47	13.97	11/26/13	17.57	07/16/14	22.45	100%
19	173.60	10.48	11/26/13	17.57	05/19/14	20.46	0%
20	130.20	7.86	07/16/14	22.44	11/23/14	23.78	100%
23	159.60	14.55	11/23/14	23.78	05/01/15	27.88	16.62%
24	159.60	14.55	11/23/14	23.78	05/01/15	27.88	16.62%
25	220.50	10.50	11/23/14	23.78	07/01/15	25.99	66.76%
27	56.00	8.40	05/06/15	27.58	07/01/15	28.83	33.24%
28	50.63	5.03	07/01/15	28.83	08/20/15	29.26	0%
29	86.80	8.63	07/16/15	29.10	10/10/15	30.35	100%
30	100.80	10.34	10/13/15	30.35	01/22/16	32.06	100%
32	14.00	4.20	01/22/16	32.06	02/05/16	32.34	100%
34	0.00	0.00	05/01/15	27.88	05/01/15	27.88	0%
35	5.60	14.45	11/21/13	10.00	11/26/13	17.57	100%
36	10.50	16.12	07/01/15	25.99	07/12/15	30.58	66.76%

By examining the CIs of the tasks in Table 11-22 and Table 11-23, we can reduce the equation representing the duration of the project to the following:

$D = D_{11} + D_{12} + D_{14} + D_{16} + D_{35} + D_{18} + D_{20} + D_{[23,28]} + D_{29} + D_{30} + D_{32}$, where

$D_{[23,28]} = max[max(D_{23}, D_{24}) + D_{27}, D_{25} + D_{36}]$.

The use of this specific relationship is restricted to the definitions of the duration PDFs defined in the model. If any of the PDFs of schedule duration changed in a manner that would affect the CIs of the tasks, the relationship may change.

The PDF of the schedule distribution can be approximated by modeling it as a lognormal distribution, however if there are discrete risks in the probabilistic critical path (i.e., CI for any discrete risk is greater than zero) the distribution is accurately modeled as a mixed distribution. Examining tasks 34, 35 and 36 we see that tasks 35 and 36 (risks R_2 and R_3, respectively) are on the probabilistic critical path, so the project schedule will have a mixed distribution. To compare the lognormal approximation to the mixed distribution, we calculate the lognormal parameters P and Q for the schedule duration in workdays then derive the percentile statistics for the total schedule duration.

Using Equations 4-5 and 4-6, with $\mu_{D'_{Tot}} = 872.88wd$ and $\sigma_{D'_{Tot}} = 23.09wd$, $P_{D'_{Tot}} = 6.7714$, and $Q_{D'_{Tot}} = 0.0265$. The resulting plot of the lognormal approximation to the total schedule duration is shown in Figure 11-9.

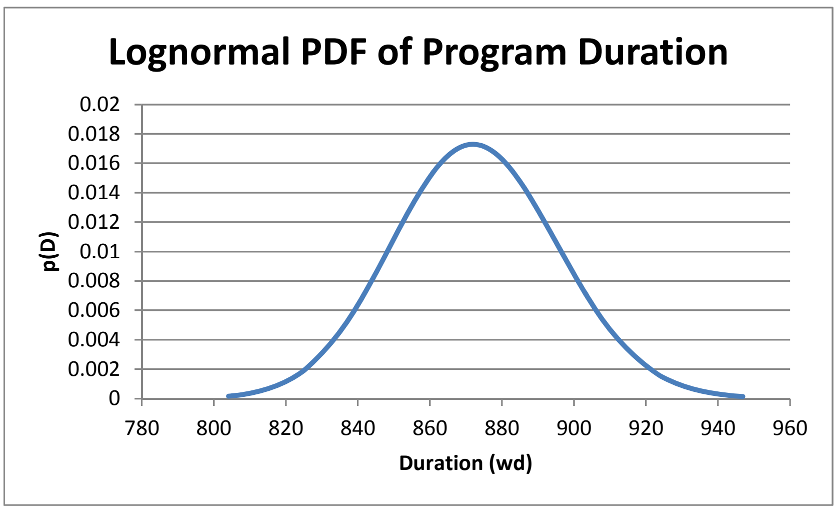

Figure 11-9 Lognormal Approximation of Total Schedule Duration

The PDF of the mixed distribution is composed of a continuous distribution consisting of tasks 11-30 that are always on the critical path (i.e., CI-100%) and combinations of state-dependent discrete risk durations. Since there are two schedule risks, we expect $2^n = 2^2 = 4$ risk states with conditional outcomes. Beginning with the risk states, S_i:

S_0 : R_2 and R_3 do not occur. $P(S_0) = (1 - 0.3)(1 - 0.15) = (0.7)(0.85) = 0.595$
S_1 : R_2 occurs and R_3 does not occur. $P(S_1) = (0.3)(1 - 0.15) = (0.3)(0.85) = 0.255$

S_2 : R_2 does not occur and R_3 occurs. $P(S_2) = (1 - 0.3)(\,0.15) = (0.7)(0.15) = 0.105$
S_3 : R_2 and R_3 occur. $P(S_3) = (0.3)(\,0.15) = 0.045$

Risk R_2 is a discrete uniform distribution with zero duration if the risk does not occur and has three equiprobable outcomes if the risk occurs ($15wd$, $25wd$, or $40wd$). The equiprobable outcomes have conditional probabilities, $P(D)|P(R_2) = 0.15/3 = 0.05$. Risk R_2 has a $CI = 1$ whether it occurs or not, so it will always be on the critical path. If risk R_3 occurs, it has a $CI{\sim}1$, but if it does not occur, its $CI = 0.5259$, and we will have to use the maximum of two PDFs to determine the correct duration to use. It has two possible outcomes: if the risk does not occur the duration is zero, and if the risk occurs the duration is modeled by a triangular distribution $T(20{,}25{,}30)wd$. Given the contingent probabilities of the possible outcomes, we have:

S_0 : 1 outcome: $P(S_0) = 0.595$, $D_{S_0} = 0wd$
S_1 : 3 outcomes:

$$P(S_{1a}) = (0.255)\left(\frac{1}{3}\right) = 0.085\ ;\ D_{S_{1a}} = 15wd$$
$$P(S_{1b}) = (0.255)\left(\frac{1}{3}\right) = 0.085\ ;\ D_{S_{1b}} = 25wd$$
$$P(S_{1c}) = (0.255)\left(\frac{1}{3}\right) = 0.085\ ;\ D_{S_{1c}} = 40wd$$

S_2 : 1 outcome: $P(S_2) = 0.105$; $D_{S_2} = T(20{,}25{,}30)wd$
S_3 : 3 outcomes:

$$P(S_{3a}) = (0.045)\left(\frac{1}{3}\right) = 0.015\ ;\ D_{S_{3a}} = 15 + T(20{,}25{,}30) = T(35{,}40{,}45)wd$$
$$P(S_{3b}) = (0.045)\left(\frac{1}{3}\right) = 0.015\ ;\ D_{S_{3b}} = 25 + T(20{,}25{,}30) = T(45{,}50{,}55)wd$$
$$P(S_{3c}) = (0.045)\left(\frac{1}{3}\right) = 0.015\ ;\ D_{S_{3c}} = 40 + T(20{,}25{,}30) = T(60{,}65{,}70)wd$$

The continuous distribution to which we combine these discrete risk states (with eight possible outcomes and associated probabilities of occurrence) is composed of tasks 11, 12, 14, 16, 18, 20, 29, and 30. All of these tasks are on the critical path 100% of the time and have uncorrelated durations, so their durations are additive. The means will be additive and the standard deviation of the total will be the square root of the sum of the squares of the standard deviations. The resulting statistics of the continuous distribution are shown in Table 11-24.

Table 11-24 Continuous Distribution Statistics

Task	$\mu_{D'}$	$\sigma_{D'}$
11	101.67	3.12
12	50.83	1.56
14	82.67	4.99
16	62.00	3.74
18	165.33	9.98
20	93.00	5.61
29	62.00	6.16
30	72.00	7.38
32	10.00	3.00
Total	699.50	16.84

When R_3 does not occur ($\overline{R_3}$), the duration of the discrete distribution is governed by the following equation:

$$D_{[23,28]} = \max(max(D_{23}, D_{24}) + D_{27}, D_{25})$$

Calculating the mean and standard deviations of the maximum of these distributions (assuming again that $\rho_{D'_A,D'_B} = 0.75$) using Equations 10-8 through 10-10, we get: $\mu_{D_{\overline{R_3}}} = 160.40wd$ and $\sigma_{D_{\overline{R_3}}} = 9.50wd$.

The resulting duration statistics for each state are shown in Table 11-25.

Table 11-25 Discrete State Duration Statistics of $D_{[23,28]}$ and D_{35}

State	Risk Occurrence	Prob	$\mu_{D'}$	$\sigma_{D'}$
S_0	$\overline{R_2} \cap \overline{R_3}$	0.595	160.40	9.50
S_{1a}	$R_{2a} \cap \overline{R_3}$	0.085	175.40	9.50
S_{1b}	$R_{2b} \cap \overline{R_3}$	0.085	185.40	9.50
S_{1c}	$R_{2b} \cap \overline{R_3}$	0.085	200.40	9.50
S_2	$\overline{R_2} \cap R_3$	0.105	182.50	9.71
S_{3a}	$R_{2a} \cap R_3$	0.015	197.50	9.71
S_{3b}	$R_{2b} \cap R_3$	0.015	207.50	9.71
S_{3c}	$R_{2c} \cap R_3$	0.015	222.50	9.71

Combining the continuous and discrete duration statistics into mixed distribution statistics (Table 11-26) allows us to compose the mixed distribution shown in Figure 11-10.

Table 11-26 Mixed Distribution of Duration Statistics

State	Risk Occurrence	Prob.	$\mu_{D'}$	$\sigma_{D'}$	$P_{D'}$	$Q_{D'}$
S_0	$\overline{R_2} \cap \overline{R_3}$	0.595	859.90	19.33	6.757	0.022
S_{1a}	$R_{2a} \cap \overline{R_3}$	0.085	874.90	19.33	6.774	0.022
S_{1b}	$R_{2b} \cap \overline{R_3}$	0.085	884.90	19.33	6.785	0.022
S_{1c}	$R_{2b} \cap \overline{R_3}$	0.085	899.90	19.33	6.802	0.021
S_2	$\overline{R_2} \cap R_3$	0.105	882.00	19.44	6.782	0.022
S_{3a}	$R_{2a} \cap R_3$	0.015	897.00	19.44	6.799	0.022
S_{3b}	$R_{2b} \cap R_3$	0.015	907.00	19.44	6.810	0.021
S_{3c}	$R_{2c} \cap R_3$	0.015	922.00	19.44	6.826	0.021

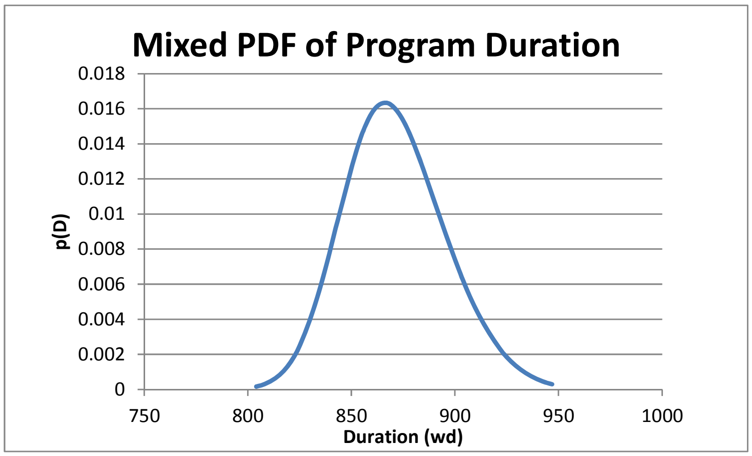

Figure 11-10 Mixed Distribution of Total Schedule Duration

When we compare plots of the lognormal approximation to the mixed distribution we see the lognormal approximation is a reasonable one.

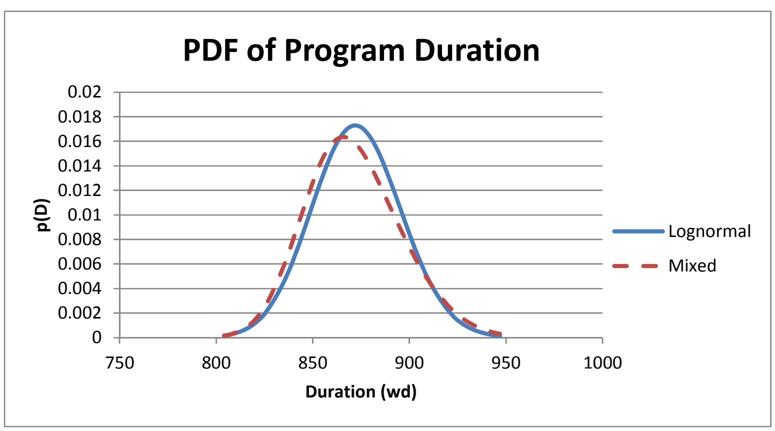

Figure 11-11 Mixed Distribution and Lognormal Approximation of Total Schedule Duration

Comparing the results of our analytic approximation to a 100,000-trial statistical simulation we see very good agreement as well. Differences in the statistics are due to sampling errors in the simulation (for wd statistics) and due to conversion of the analytic results into calendar dates (for cd statistics).

Table 11-27 Comparison of Analytic and Statistical Simulation Finish Date Statistics

Finish Date	Analytic Approach	Statistical Simulation
$\mu_{F'}$ (wd)	02/20/15	02/18/15
$\sigma_{F'}$ (wd)	23.09	23.74
μ_F (cd)	02/05/16	01/24/16
σ_F (cd)	32.34	33.17

11.2.2 Cost Probability Distribution

The program's costs are the sum of the lowest-level WBS elements shown in Table 11-28. The cost of each lowest-level WBS element is defined by a time-dependent (TD) costs (i.e., those costs that vary with the duration of each task), and a time-independent (TI) cost (i.e., the probabilistic daily rate or other additive costs not related to schedule duration).

Table 11-28 NASA Example WBS and Point Estimate

WBS	WBS Description	Point Estimate, $
1	Analysis File	$151,500,000.00
1.1	Milestone Summary	$0.00
1.1.1	Project ATP	
1.1.2	PDR	
1.1.3	CDR	
1.1.4	Rocket delivery	
1.2	Project Support Costs hammock task	$20,000,000.00
1.2.1	Support Start	
1.2.2	Support Finish	
1.3	Preliminary Design	$9,000,000.00
1.3.1	Requirements definition and documentation	$4,000,000.00
1.3.2	Preliminary design activities	$5,000,000.00
1.4	Detailed Design	$48,500,000.00
1.4.1	Initial detailed design	
1.4.2	Design GN&C	$15,000,000.00
1.4.3	Trade studies and analysis	
1.4.4	Design pyrotechnics	$7,500,000.00
1.4.5	Design propulsion system	$12,000,000.00
1.4.6	Design structures and mechanisms	$9,000,000.00
1.4.7	Finalize integrated design	$5,000,000.00
1.5	Development and Unit Testing	$42,000,000.00
1.5.1	Fabricate rocket Components	$30,000,000.00
1.5.1.1	Fabricate and unit test structure (including pyros)	$20,000,000.00
1.5.1.2	Fabricate and unit test engine	$10,000,000.00
1.5.2	Develop and test flight software for GN&C	$12,000,000.00
1.6	Integration and Testing	$29,000,000.00
1.6.1	Integrate rocket components	$6,000,000.00

WBS	WBS Description	Point Estimate, $
1.6.2	Test frame, fuel system and engine	$8,000,000.00
1.6.3	Test guidance system	$5,000,000.00
1.6.4	Final integration and testing	$10,000,000.00
1.7	Delivery	$3,000,000.00
1.7.1	Delivery	$3,000,000.00
2	Risk Register	$0.00
2.1	Risk 1 - TI - Additional Purchase	$0.00
2.2	Risk 2 - Duration - Additional Studies Required	$0.00
2.3	Risk 3 - TI and Duration - Delay from Additional Software Purchase	$0.00

Individual lowest-level WBS element Costs, X_i, are defined by the combination of TD and TI costs as follows:

$$X_i = \begin{cases} (TD_i \varepsilon_{TD_i})(TI_i \varepsilon_{TI_i}) = Duration'_i \varepsilon_{TD_i} Rate_i \varepsilon_{TI_i} & , if\ TI\ is\ multiplicative \\ [(TD_i \varepsilon_{TD_i})(TI_i)] + \varepsilon_i = (Duration'_i \varepsilon_{TD_i} Rate_i) + \varepsilon_{TI_i} & , if\ TI\ is\ additive \end{cases}, \quad \textbf{11-2}$$

where:

ε_{TI_i} is the TI PDF

ε_{TD_i} is the TD PDF

$Duration'_i$ is the probabilistic task duration in wd.

$Rate_i$ is the nominal cost per wd.

11.2.2.1 Cost-Estimating-Level Uncertainty Statistics

The rate, and the TI and TD PDFs for each lowest-level WBS element in the NASA example are shown in Table 11-29.

Table 11-29 NASA Resource-Loaded Schedule TI and TD Cost PDFs

WBS	Rate ($/wd.)	TD Cost PDF	TI Cost PDF
1.2	$23,809.52	N*(100,5)	
1.3.1	$40,000.00	T*(95,100,105)	
1.3.2	$90,000.00	T*(95,100,105)	N (500000,40000); ρ (DESFABCOST=0.3)
1.4.1	$0.00		
1.4.2	$93,750.00	T*(95,100,105)	
1.4.3			
1.4.4	$75,000.00	T*(95,100,105)	
1.4.5	$75,000.00	T*(95,100,105)	
1.4.6	$75,000.00	T*(95,100,105)	
1.4.7	$55,555.56	T*(95,100,105)	
1.5.1.1	$166,666.67	T*(95,100,105)	T*(80,100,110); ρ(DESFABCOST=0.3)
1.5.1.2	$83,333.33	T*(95,100,105)	T*(80,100,110); ρ (DESFABCOST=0.3)
1.5.2	$80,000.00	T*(95,100,105)	

1.6.1	$150,000.00	T*(95,100,105)	
1.6.2	$228,571.43	T*(95,100,105)	
1.6.3	$83,333.33	T*(95,100,105)	
1.6.4	$142,857.14	T*(95,100,105)	
1.7.1	$300,000.00		
2.1			R(0.3,T($8M,$10M,$13M))
2.2			
2.3			R(0.3,T($13M,$15M,$20M))

The means and standard deviations of the triangular TD and TI PDFs are calculated using Equations 4-1and 4-2. Table 11-30 shows the rates and workday duration statistics of the schedule summary tasks.

Table 11-30 Rate, Duration and Uncertainty Statistics for Cost-Estimating-Level WBS Elements

WBS	Rate	$\mu_{D'}$	$\sigma_{D'}$	$\mu_{\varepsilon_{TD}}$	$\sigma_{\varepsilon_{TD}}$	$\mu_{\varepsilon_{TI}}$	$\sigma_{\varepsilon_{TI}}$
1.2	$23,809.52	872.88	23.10	1	0.0500		
1.3.1	$40,000.00	101.67	3.12	1	0.0204		
1.3.2	$90,000.00	50.83	1.56	1	0.0204	+500000	40000
1.4.2	$93,750.00	165.33	9.98	1	0.0204		
1.4.4	$75,000.00	103.33	6.24	1	0.0204		
1.4.5	$75,000.00	165.33	9.98	1	0.0204		
1.4.6	$75,000.00	124.00	7.48	1	0.0204		
1.4.7	$55,555.56	93.00	5.61	1	0.0204		
1.5.1.1	$166,666.67	114.00	10.39	1	0.0204	0.9667	0.0624
1.5.1.2	$83,333.33	114.00	10.39	1	0.0204	0.9667	0.0624
1.5.2	$80,000.00	157.5	7.50	1	0.0204	1	0
1.6.1	$150,000.00	40.00	6.00	1	0.0204		
1.6.2	$228,571.43	36.17	3.60	1	0.0204		
1.6.3	$83,333.33	62.00	6.16	1	0.0204		
1.6.4	$142,857.14	72.00	7.38	1	0.0204		
1.7.1	$300,000.00	10.00	3.00	1	0		

Using values form Table 11-30 and Equation 11-2, we can calculate the mean and standard deviation of each cost-estimating-level WBS Element (Table 11-31).

Table 11-31 Mean and Standard Deviations of Cost-Estimating-Level WBS Elements

WBS	μ_X	σ_X
1.2	$20,782,813.74	$549,947.19
1.3.1	$4,066,666.67	$149,842.51
1.3.2	$5,075,000.00	$173,253.56
1.4.2	$15,500,000.00	$987,658.22
1.4.4	$7,750,000.00	$493,829.11
1.4.5	$12,400,000.00	$790,126.57
1.4.6	$9,300,000.00	$592,594.93

WBS	μ_X	σ_X
1.4.7	$5,166,666.67	$329,219.41
1.5.1.1	$18,366,666.67	$2,088,349.29
1.5.1.2	$9,183,333.33	$1,044,174.65
1.5.2	$12,600,000.00	$652,916.53
1.6.1	$6,000,000.00	$908,480.87
1.6.2	$8,266,666.67	$839,232.45
1.6.3	$5,166,666.67	$524,520.28
1.6.4	$10,285,714.29	$1,075,541.71
1.7.1	$3,000,000.00	$900,000.00

11.2.2.2 Computing WBS-Element Correlations

The statistics of the summary-level WBS elements are computed using the FRISK method described in Section 4.2.2.1. All but four of the WBS elements in the NASA resource-loaded schedule are uncorrelated to each other. Correlations are defined between the following: 1) schedule duration PDFs for WBS elements 1.5.1.1, 1.5.1.2, and 1.5.2 (i.e., tasks 23, 24 and 25) with a correlation coefficient defined by $\rho_{DEVDUR} = 0.75$; and between time independent cost PDFs for WBS elements 1.3.2, 1.5.1.1, 1.5.1.2 (i.e., tasks 12, 23 and 24) with a correlation coefficient defined by $\rho_{DESFABCOST} = 0.3$.

The effects of the correlated schedule durations will manifest themselves in the standard deviations of the cost summations of WBS elements 1.5 and 1.51. The correlated time independent cost correlations will affect the standard deviations of the WBS elements where they are summed (i.e. WBS elements 1 and 1.51). The standard deviations of all other summary WBS elements can be computed using a root-sum-square of their constituent WBS elements.

The correlations between schedule durations and the respective costs for WBS elements 1.5.1.1, 1.5.1.2, and 1.5.2 are $\rho_{1.5.1.1,1.5.1.2}$, $\rho_{1.5.1.1,1.5.2}$, and $\rho_{1.5.1.2,1.5.2}$, respectively. We will calculate them in that order.

$$\boldsymbol{\rho_{1.5.1.1,1.5.1.2}}$$

We use Equation 4-26, and following the steps in Section 8 to compute the correlation coefficient. From Section 8, Step 1, which is:

$$\rho_{1.5.1.1,1.5.1.2} = \frac{E[X_{1.5.1.1}X_{1.5.1.2}] - E[X_{1.5.1.1}]E[X_{1.5.1.2}]}{\sigma_{1.5.1.1}\sigma_{1.5.1.2}} = \frac{E[X_{1.5.1.1}X_{1.5.1.2}] - \mu_{1.5.1.1}\mu_{1.5.1.2}}{\sigma_{1.5.1.1}\sigma_{1.5.1.2}}$$

$$X_{1.5.1.1} = \left(Duration'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\right)\left(Rate_{1.5.1.1}\varepsilon_{TI_{1.5.1.1}}\right)$$

The TD uncertainty defined for WBS 1.5.1.1 is $\varepsilon_{TD_{1.5.1.1}} = T(0.95,1.00,1.05)$. Using Equations 4-1 and 4-2, $\mu_{\varepsilon_{TD_{1.5.1.1}}} = 1$, and $\sigma_{\varepsilon_{TD_{1.5.1.1}}} = 0.0204$.

The TI uncertainty defined for WBS 1.5.1.1 is $\varepsilon_{TI_{1.5.1.1}} = T(0.80{,}1.00{,}1.10)$. Using Equations 4-1 and 4-2 we get: $\mu_{\varepsilon_{TI_{1.5.1.1}}} = 0.9667$, and $\sigma_{\varepsilon_{TI_{1.5.1.1}}} = 0.0624$.

We can rearrange the cost function as $X_{1.5.1.1} = R_{1.5.1.1}{D'}_{1.5.1.1}\varepsilon_{TD(1.5.1.1)}\varepsilon_{TI(1.5.1.1)}$, and by setting $\varepsilon_{1.5.1.1} = \varepsilon_{TD(1.5.1.1)}\varepsilon_{TI(1.5.1.1)}$, we can simplify some of the equations.

By definition for each WBS element, the TI and TD uncertainty PDFs are uncorrelated, so

$\mu_{\varepsilon_{1.5.1.1}} = \mu_{\varepsilon_{TD(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.1)}}$, and

$$\sigma_{\varepsilon_{1.5.1.1}} = \sqrt{\left(\sigma_{\varepsilon_{TD(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.1)}}\right)^2 + \left(\mu_{\varepsilon_{TD(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\right)^2 + \left(\sigma_{\varepsilon_{TD(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\right)^2}.$$

From Table 11-30 $\mu_{{D'}_{1.5.1.1}} = \mu_{{D'}_{23}} = 114wd$, $\sigma_{{D'}_{1.5.1.1}} = \sigma_{{D'}_{23}} = 10.39wd$, and $Rate_{1.5.1.1} = \$166{,}666.67$, which is a constant.

From Step 2a, $\mu_{1.5.1.1} = \mu_{R_{1.5.1.1}}\left(\mu_{{D'}_{1.5.1.1}}\mu_{\varepsilon_{1.5.1.1}}\right) = \$166{,}666.67(114)(0.9667) = \$18{,}366{,}666.67$

From Step 2b, $\sigma_{1.5.1.1} = Var(\sigma_{1.5.1.1}) = R_{1.5.1.1}Var({D'}_{1.5.1.1}\varepsilon_{1.5.1.1})$

Using the propagation of errors method:

$$\sigma_{1.5.1.1} = R_{1.5.1.1}\sqrt{\left(\sigma_{{D'}_{1.5.1.1}}\mu_{\varepsilon_{1.5.1.1}}\right)^2 + \left(\mu_{{D'}_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.1}}\right)^2 + \left(\sigma_{{D'}_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.1}}\right)^2}$$

$$\sigma_{1.5.1.1} = \$166{,}666.67\sqrt{([10.39][0.9667])^2 + ([114][0.06542])^2 + ([10.39][0.06542])^2}$$

$\sigma_{1.5.1.1} = \$2{,}088{,}349.29$.

Using the same formulation for $X_{1.5.1.2}$, we get:

$\mu_{1.5.1.2} = \$9{,}183{,}333.33$ and $\sigma_{1.5.1.2} = \$1{,}044{,}174.65$

From step 2c,

$$X_{1.5.1.1}X_{1.5.1.2} = R_{1.5.1.1}({D'}_{1.5.1.1}\varepsilon_{1.5.1.1})R_{1.5.1.2}({D'}_{1.5.1.2}\varepsilon_{1.5.1.2})$$

$$X_{1.5.1.1}X_{1.5.1.2} = R_{1.5.1.1}R_{1.5.1.2}({D'}_{1.5.1.1}\varepsilon_{1.5.1.1})({D'}_{1.5.1.2}\varepsilon_{1.5.1.2})$$

From Step 2d, $a = R_{1.5.1.1}R_{1.5.1.2}$

$$E[X_{1.5.1.1}X_{1.5.1.2}] = aE[({D'}_{1.5.1.1}\varepsilon_{1.5.1.1})({D'}_{1.5.1.2}\varepsilon_{1.5.1.2})]$$

Grouping correlated error terms gives us:

$E[X_{1.5.1.1}X_{1.5.1.2}] = aE[(D'_{1.5.1.1}D'_{1.5.1.2})(\varepsilon_{1.5.1.1}\varepsilon_{1.5.1.2})]$

$E[X_{1.5.1.1}X_{1.5.1.2}] = aE[(D'_{1.5.1.1}D'_{1.5.1.2})]E[(\varepsilon_{1.5.1.1}\varepsilon_{1.5.1.2})]$

$E[D'_{1.5.1.1}D'_{1.5.1.2}] = \mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.1.2}} + \rho_{D'_{1.5.1.1},D'_{1.5.1.2}}\sigma_{D'_{1.5.1.1}}\sigma_{D'_{1.5.1.2}}$

$E[D'_{1.5.1.1}D'_{1.5.1.2}] = (114)(114) + (0.75)(10.39)(10.39) = 13{,}077$

Expanding the expectation of the uncertainty term, we get:

$E[\varepsilon_{1.5.1.2}\varepsilon_{1.5.1.1}] = E\left[\varepsilon_{TD(1.5.1.1)}\varepsilon_{TI(1.5.1.1)}\varepsilon_{TD(1.5.1.2)}\varepsilon_{TI(1.5.1.2)}\right]$

$= E\left[\varepsilon_{TD(1.5.1.1)}\varepsilon_{TD(1.5.1.2)}\right]E\left[\varepsilon_{TI(1.5.1.1)}\varepsilon_{TI(1.5.1.2)}\right]$

$= \left(\mu_{\varepsilon_{TD(1.5.1.1)}}\mu_{\varepsilon_{TD(1.5.1.2)}}\right)\left(\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.2)}} + \rho_{\varepsilon_{TI(1.5.1.1)},\varepsilon_{TI(1.5.1.2)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.2)}}\right)$

$= \left(\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.2)}} + \rho_{\varepsilon_{TI(1.5.1.1)},\varepsilon_{TI(1.5.1.2)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.2)}}\right)$

$E[\varepsilon_{1.5.1.2}\varepsilon_{1.5.1.1}] = \mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.2)}} + \rho_{\varepsilon_{TI(1.5.1.1)},\varepsilon_{TI(1.5.1.2)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.2)}}$

$E[\varepsilon_{1.5.1.2}\varepsilon_{1.5.1.1}] = (0.9667)(0.9667) + (0.3)(0.0624)(0.0624) = 0.9356$

Recombining terms, we get:

$E[D'_{1.5.1.1}D'_{1.5.1.2}]E[\varepsilon_{1.5.1.2}\varepsilon_{1.5.1.1}] =$
$\left(\mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.1.2}} + \rho_{D'_{1.5.1.1},D'_{1.5.1.2}}\sigma_{D'_{1.5.1.1}}\sigma_{D'_{1.5.1.2}}\right)\left(\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.1.2)}} + \right.$
$\left.\rho_{\varepsilon_{TI(1.5.1.1)},\varepsilon_{TI(1.5.1.2)}}\sigma_{\varepsilon_{TI(1.5.1.1)}}\sigma_{\varepsilon_{TI(1.5.1.2)}}\right)$

$\frac{E[X_{1.5.1.1}X_{1.5.1.2}]}{a} = (13077)(0.9356\ ) = 12{,}234.99$

$\mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.1.2}} = (114)(114) = 12{,}996$

From Step 3, and removing the rate term, we get:

$\rho_{1.5.1.1,1.5.1.2} = \frac{12{,}234.99-12{,}996}{(10.6518)(10.6518)} = 0.5793$

$\boldsymbol{\rho_{1.5.1.1,1.5.2}}$ and $\boldsymbol{\rho_{1.5.1.2,1.5.2}}$

$\rho_{1.5.1.1,1.5.2}$ and $\rho_{1.5.1.2,1.5.2}$ are calculated in a similar fashion, except $\rho_{\varepsilon_{1.5.11},\varepsilon_{1.5.2}} = 0$ and $\rho_{\varepsilon_{1.5.12},\varepsilon_{1.5.2}} = 0$.

This simplifies the product moment term to:

$E[D'_{1.5.1.1}D'_{1.5.2}]E[\varepsilon_{1.5.2}\varepsilon_{1.5.1.1}] =$
$\left(\mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.2}} + \rho_{D'_{1.5.1.1},D'_{1.5.2}}\sigma_{D'_{1.5.1.1}}\sigma_{D'_{1.5.2}}\right)\left(\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.2)}}\right)$

This results in correlation equations $\rho_{1.5.1.1,1.5.2}$ and $\rho_{1.5.1.2,1.5.2}$ (by similarity), which are:

$$\rho_{1.5.1.1,1.5.2} = \frac{\left(\mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.2}}+\rho_{D'_{1.5.1.1},D'_{1.5.2}}\sigma_{D'_{1.5.1.1}}\sigma_{D'_{1.5.2}}\right)\left(\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.2)}}\right)-\mu_{D'_{1.5.1.1}}\mu_{D'_{1.5.2}}}{\sqrt{\left(\sigma_{D'_{1.5.1.1}}\right)^2+\left(\mu_{D'_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.1}}\right)^2+\left(\sigma_{D'_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.1}}\right)^2}\sqrt{\left(\sigma_{D'_{1.5.1.3}}\right)^2+\left(\mu_{D'_{1.5.1.3}}\sigma_{\varepsilon_{1.5.1.3}}\right)^2+\left(\sigma_{D'_{1.5.1.3}}\sigma_{\varepsilon_{1.5.1.3}}\right)^2}}$$

$$\rho_{1.5.1.2,1.5.2} = \frac{\left(\mu_{D'_{1.5.1.2}}\mu_{D'_{1.5.2}}+\rho_{D'_{1.5.1.2},D'_{1.5.2}}\sigma_{D'_{1.5.1.2}}\sigma_{D'_{1.5.2}}\right)\left(\mu_{\varepsilon_{TI(1.5.1.2)}}\mu_{\varepsilon_{TI(1.5.2)}}\right)-\mu_{D'_{1.5.1.2}}\mu_{D'_{1.5.2}}}{\sqrt{\left(\sigma_{D'_{1.5.1.2}}\right)^2+\left(\mu_{D'_{1.5.1.2}}\sigma_{\varepsilon_{1.5.1.2}}\right)^2+\left(\sigma_{D'_{1.5.1.2}}\sigma_{\varepsilon_{1.5.1.2}}\right)^2}\sqrt{\left(\sigma_{D'_{1.5.2}}\right)^2+\left(\mu_{D'_{1.5.2}}\sigma_{\varepsilon_{1.5.2}}\right)^2+\left(\sigma_{D'_{1.5.2}}\sigma_{\varepsilon_{1.5.2}}\right)^2}}$$

Solving $\rho_{1.5.1.2,1.5.2}$ using the parameters from Table 11-30, we get:

$$\mu_{D'_{1.5.1.2}}\mu_{D'_{1.5.2}} = (114)(157.5) = 17{,}955$$

$$\rho_{D'_{1.5.1.2},D'_{1.5.2}}\sigma_{D'_{1.5.1.2}}\sigma_{D'_{1.5.2}} = (0.75)(10.39)(7.50) = 58.46$$

$$\mu_{\varepsilon_{TI(1.5.1.1)}}\mu_{\varepsilon_{TI(1.5.2)}} = (0.9667)(1) = 0.9667$$

$$\sqrt{\left(\sigma_{D'_{1.5.1.1}}\right)^2 + \left(\mu_{D'_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.2}}\right)^2 + \left(\sigma_{D'_{1.5.1.1}}\sigma_{\varepsilon_{1.5.1.2}}\right)^2} = 10.65wd$$

$$\sqrt{\left(\sigma_{D'_{1.5.2}}\right)^2 + \left(\mu_{D'_{1.5.2}}\sigma_{\varepsilon_{1.5.2}}\right)^2 + \left(\sigma_{D'_{1.5.2}}\sigma_{\varepsilon_{1.5.2}}\right)^2} = 8.16wd, \text{ so}$$

$$\rho_{1.5.1.1,1.5.2} = \frac{(17{,}955+58.46)(0.9667)-17{,}955}{(10.65)(8.16)} = 0.5526$$

Coincidently, the values for $\rho_{1.5.1.2,1.5.2}$ are the same, so

$$\rho_{1.5.1.2,1.5.2} = \frac{(17{,}955+58.46)(0.9667)-17{,}955}{(10.65)(8.16)} = 0.5526\ .$$

The correlation matrix for WBS 1.5's subordinate elements is:

$$\boldsymbol{\rho}_{1.5} = \begin{bmatrix} 1 & 0.5793 & 0.5526 \\ 0.5793 & 1 & 0.5526 \\ 0.5526 & 0.5526 & 1 \end{bmatrix}$$

$\boldsymbol{\rho_{1.3.2,1.5.1.1}}$, and $\boldsymbol{\rho_{1.3.2,1.5.2}}$

The second set of correlations defined in the NASA resource-loaded schedule are those defined between TI PDFs. The correlations between independent cost PDFs affect the correlation between WBS elements 1.3.2, 1.5.1.1, 1.5.1.2. We need to calculate $\rho_{1.3.2,1.5.1.1}$, and $\rho_{1.3.2,1.5.2}$. Since there is no correlation between the durations of these WBS elements,

$$\rho_{1.3.2,1.5.1.1} = \frac{E[X_{1.3.2}X_{1.5.1.1}] - E[X_{1.3.2}]E[X_{1.5.1.1}]}{\sigma_{1.3.2}\sigma_{1.5.1.2}} = \frac{E[X_{1.5.1.1}X_{1.5.1.2}] - \mu_{1.3.2}\mu_{1.5.1.1}}{\sigma_{1.3.2}\sigma_{1.5.1.1}}$$

$$X_{1.3.2} = \left(D'_{1.3.2}\varepsilon_{TD_{1.3.2}}\right)(R_{1.3.2}) + \varepsilon_{TI_{1.3.2}}$$

$$X_{1.5.1.1} = \left(D'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\right)\left(R_{1.5.1.1}\varepsilon_{TI_{1.5.1.1}}\right)$$

$$X_{1.3.2}X_{1.5.1.1} = \left[\left(D'_{1.3.2}\varepsilon_{TD_{1.3.2}}\right)(R_{1.3.2}) + \varepsilon_{TI_{1.3.2}}\right]\left(D'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\right)\left(R_{1.5.1.1}\varepsilon_{TI_{1.5.1.1}}\right)$$

$$X_{1.3.2}X_{1.5.1.1} = R_{1.3.2}R_{1.5.1.1}D'_{1.3.2}D'_{1.5.1.1}\varepsilon_{TD_{1.3.2}}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}} + \varepsilon_{TI_{1.3.2}}R_{1.5.1.1}D'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}}$$

Setting $a = R_{1.3.2}R_{1.5.1.1}$ (a constant) we get:

$$X_{1.3.2}X_{1.5.1.1} = aD'_{1.3.2}D'_{1.5.1.1}\varepsilon_{TD_{1.3.2}}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}} + \varepsilon_{TI_{1.3.2}}R_{1.5.1.1}D'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}}$$

$$E[X_{1.3.2}X_{1.5.1.1}] = aE\left[D'_{1.3.2}D'_{1.5.1.1}\varepsilon_{TD_{1.3.2}}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}}\right] + R_{1.5.1.1}E\left[D'_{1.5.1.1}\varepsilon_{TD_{1.5.1.1}}\varepsilon_{TI_{1.5.1.1}}\varepsilon_{TI_{1.3.2}}\right]$$

Separating the correlated terms results in:

$$E[X_{1.3.2}X_{1.5.1.1}] = aE[D'_{1.3.2}D'_{1.5.1.1}]E\left[\varepsilon_{TD_{1.3.2}}\varepsilon_{TD_{1.5.1.1}}\right]E\left[\varepsilon_{TI_{1.5.1.1}}\right] + R_{1.5.1.1}E[D'_{1.5.1.1}]E\left[\varepsilon_{TD_{1.5.1.1}}\right]E\left[\varepsilon_{TI_{1.5.1.1}}\varepsilon_{TI_{1.3.2}}\right]$$

$\mu_{1.3.2} = \$5{,}075{,}000.00$ and $\sigma_{1.3.2} = \$173{,}253.56$

$\mu_{1.5.1.1} = \$18{,}366{,}666.67$ and $\sigma_{1.5.1.1} = \$2{,}088{,}349.29$

Computing each product moment term

$$a = R_{1.3.2}R_{1.5.1.1} = (\$90{,}000.00)(\$166{,}666.67) = 1.5E + 10$$

$E[D'_{1.5.1.1}] = 114$, and $E\left[\varepsilon_{TD_{1.5.1.1}}\right] = 1$

Since $\rho_{D'_{1.3.2},D'_{1.5.1.1}} = 0$,

$$E[D'_{1.3.2}D'_{1.5.1.1}] = \mu_{D'_{1.3.2}}\mu_{D'_{1.5.1.1}}$$

$$E[D'_{1.3.2}D'_{1.5.1.1}] = (50.83)(114) = 5{,}795$$

$E\left[\varepsilon_{TD_{1.3.2}}\varepsilon_{TD_{1.5.1.1}}\right] = \mu_{\varepsilon_{TD_{1.3.2}}}\mu_{\varepsilon_{TD_{1.5.1.1}}} = 1$

$E\left[\varepsilon_{TI_{1.5.1.1}}\right] = 0.9667$

$E\left[\varepsilon_{TI_{1.5.1.1}}\varepsilon_{TI_{1.3.2}}\right] = \mu_{\varepsilon_{TI_{1.3.2}}}\mu_{\varepsilon_{TI_{1.5.1.1}}} + \rho_{\varepsilon_{TI_{1.3.2},,TI_{1.3.2}}}\sigma_{\varepsilon_{TI_{1.3.2}}}\sigma_{\varepsilon_{TI_{1.5.1.1}}}$

Using values from Table 11-30, we get:

$E\left[\varepsilon_{TI_{1.5.1.1}}\varepsilon_{TI_{1.3.2}}\right] = (500{,}000)(0.9667) + (0.3)(40{,}000)(0.0624) = 484{,}081.66$

Computing the product moment term using previously computed values results in:

$E[X_{1.3.2}X_{1.5.1.1}] = (1.5E + 10)(5{,}795)(0.9667) + (\$166{,}666.67\ )(114)(484{,}081.66)$

$E[X_{1.3.2}X_{1.5.1.1}] = 9.32251E + 13$

Computing the product of the means provides:

$\mu_{1.3.2}\mu_{1.5.1.1} = (\$5{,}075{,}000.00)(\$18{,}366{,}666.67\ ) = 9.32108E + 13$

$E[X_{1.5.1.1}X_{1.5.1.2}] - \mu_{1.3.2}\mu_{1.5.1.1} = \$14{,}218{,}298{,}069.73$

$\sigma_{1.3.2}\sigma_{1.5.1.1} = 3.61814E + 11$

$\rho_{1.3.2,1.5.1.1} = \frac{(9.32251E+13)-(9.32108E+13)}{3.61814E+11} = 0.0393$

Substituting the values from WBS 1.5.1.2 into the equation and solving we obtain $\rho_{1.3.2,1.5.1.2} = 0.0393$. The results of a 100,000-trial statistical simulation show $\rho_{1.3.2,1.5.1.2} = 0.0389$, an excellent agreement.

11.2.2.3 Statistical Summation of WBS Element Costs

Once the correlation coefficients between correlated WBS elements have been computed, the total cost can be calculated through statistical summation. The mean of total cost from Equation 4-10 is:

$\mu_T = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \mu_{X_i}$

A simplified equation for calculating the variance of the total cost when dealing with the standard deviations of correlated (σ_{cor}) and uncorrelated (σ_{unc}) WBS elements based on Equations 4-11 and 9-12 is Equation 11-3. This relationship greatly simplifies the computation of variances of programs with many WBS elements by limiting the number of matrix multiplications required.

$\sigma_T{}^2 = \mathrm{Var}(X_T) = \boldsymbol{\sigma}_{\mathbf{unc}}{}^{\boldsymbol{T}}\boldsymbol{I}\boldsymbol{\sigma}_{\mathbf{unc}} + \boldsymbol{\sigma}_{\mathbf{cor}}{}^{\boldsymbol{T}}\boldsymbol{\rho}\boldsymbol{\sigma}_{\mathbf{cor}}$, where **11-3**
$\boldsymbol{\sigma}_{\mathbf{unc}}$ is a column vector of standard deviations of uncorrelated WBS

elements with dimension *1*x*M*,
$\boldsymbol{\sigma}_{\mathbf{cor}}$ is a column vector of standard deviations of correlated WBS elements with dimension *1*x*N*,
$\boldsymbol{I}$ is the identity matrix with dimension *M*x*M*,
$\boldsymbol{\rho}$ is the correlation matrix with dimension *N*x*N*, and
$(\quad)^{\boldsymbol{T}}$ is the transpose operation

We use the latter equation to account for the correlation between WBS elements 1.3.2, 1.5.1.1, 1.5.1.2, and 1.5.2, whose correlation matrix is (in that row and column order):

$$\boldsymbol{\rho} = \begin{bmatrix} \mathbf{1} & 0.0393 & 0.0393 & 0.0000 \\ 0.0393 & \mathbf{1} & 0.5793 & 0.5526 \\ 0.0393 & 0.5793 & \mathbf{1} & 0.5526 \\ 0.0000 & 0.5526 & 0.5526 & \mathbf{1} \end{bmatrix}$$

The results of the MOM and 100,000-trial Statistical Simulation Summation of the WBS Elements are shown in Table 11-32. These results indicate very good agreement between the two methods. Discrepancies in the results obtained using the two approaches are primarily caused by approximations used in the calculation of workday statistics using the analytic method, inexact statistical sampling of correlated random variables by the statistical simulation, and difficulties of the statistical simulation when dealing with discrete risks (as discussed in Section 9.1.8).

Table 11-32 Results of MOM and Statistical Simulation Summation of WBS Elements

WBS	Analytic Method		100,000-Trial Statistical Simulation	
	μ_X	σ_X	μ_X	σ_X
1	$160,810,194.69	$11,333,411.24	$160,756,897.76	$10,050,372.90
1.2	$20,782,813.74	$1,176,015.04	$20,730,787.20	$1,179,300.81
1.3	$9,141,666.67	$229,062.38	$9,141,657.73	$228,767.50
1.3.1	$4,066,666.67	$149,842.51	$4,066,668.80	$149,839.56
1.3.2	$5,075,000.00	$173,253.56	$5,074,988.93	$173,027.10
1.4	$50,116,666.67	$1,517,626.47	$50,116,585.59	$1,514,678.61
1.4.1	$0.00	$0.00	$0.00	$0.00
1.4.2	$15,500,000.00	$987,658.22	$15,499,948.07	$986,900.52
1.4.3	$0.00	$0.00	$0.00	$0.00
1.4.4	$7,750,000.00	$493,829.11	$7,750,025.08	$494,218.56
1.4.5	$12,400,000.00	$790,126.57	$12,399,988.36	$789,913.58
1.4.6	$9,300,000.00	$592,594.93	$9,299,952.17	$591,903.15
1.4.7	$5,166,666.67	$329,219.41	$5,166,671.91	$329,327.90
1.5	$40,150,000.00	$3,495,228.26	$40,151,376.12	$3,276,044.29
1.5.1	$27,550,000.00	$3,265,642.58	$27,551,339.35	$2,834,945.88
1.5.1.1	$18,366,666.67	$2,088,349.29	$18,367,694.56	$2,097,256.41

WBS	Analytic Method		100,000-Trial Statistical Simulation	
	μ_X	σ_X	μ_X	σ_X
1.5.1.2	$9,183,333.33	$1,044,174.65	$9,183,644.79	$1,046,930.09
1.5.2	$12,600,000.00	$652,916.53	$12,600,036.77	$653,556.59
1.6	$29,719,047.62	$1,720,918.39	$29,718,882.73	$1,724,308.57
1.6.1	$6,000,000.00	$908,480.87	$5,999,968.14	$908,846.37
1.6.2	$8,266,666.67	$839,232.45	$8,266,635.33	$838,788.65
1.6.3	$5,166,666.67	$524,520.28	$5,166,665.32	$524,447.49
1.6.4	$10,285,714.29	$1,075,541.71	$10,285,613.94	$1,074,641.82
1.7	$3,000,000.00	$900,000.00	$2,999,969.14	$900,141.94
1.7.1	$3,000,000.00	$900,000.00	$2,999,969.14	$900,141.94
2	$7,900,000.00	$10,478,546.08	$7,897,639.25	$8,785,656.92
2.1	$3,100,000.00	$5,687,706.04	$3,098,858.74	$4,766,730.77
2.2	$0.00	$0.00	$0.00	$0.00
2.3	$4,800,000.00	$8,800,564.07	$4,798,780.51	$7,374,605.76

11.2.2.4 PDF of Total Cost

The PDF of the total cost can be approximated by a lognormal distribution or by computing the exact, mixed distribution. The lognormal approximation is easily obtained, as it was for the schedule PDF, by computing the lognormal parameters P and Q then deriving the percentile statistics for total cost. Using Equations 4-5 and 4-6, with $\mu_{X_{Tot}} =$ \$160,810,256.90 and $\sigma_{X_{Tot}} = \$9,765,611.10$, $P_{X_{Tot}} = 18.8939$, and $Q_{X_{Tot}} = 0.0607$. The resulting plot of the lognormal approximation to the total schedule duration is shown in Figure 11-12.

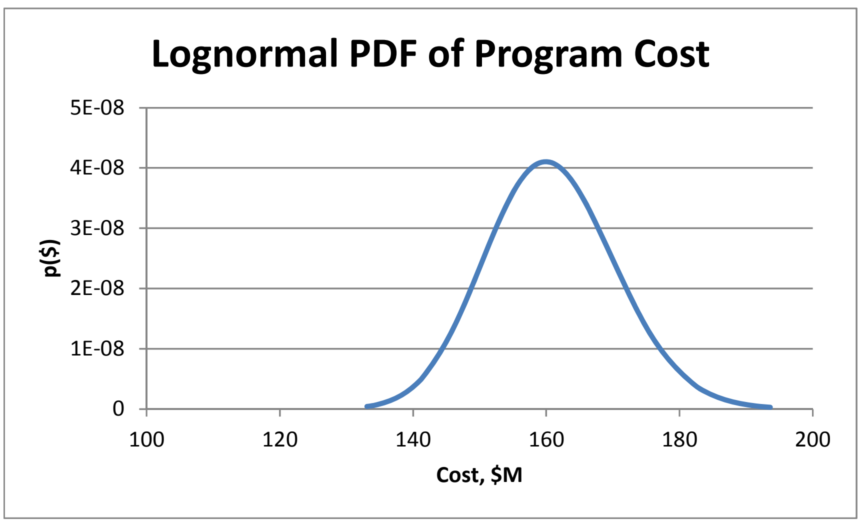

Figure 11-12 Lognormal Approximation of Total Cost

The PDF of the mixed distribution has a continuous distribution component and a discrete-risks component. Since there are two risks in the discrete-risks component affecting the total cost (R_1 and R_3), we will derive a set of risk-state statistics for each state. There are $2^n = 2^2 = 4$ risk states with conditional outcomes. Beginning with the risk states, S_i:

S_0 : R_1 and R_3 do not occur. $P(S_0) = (1 - 0.3)(1 - 0.3) = (0.7)(0.7) = 0.49$
S_1 : R_1 occurs and R_3 does not occur. $P(S_1) = (0.3)(1 - 0.3) = (0.3)(0.7) = 0.21$
S_2 : R_1 does not occur and R_3 occurs. $P(S_2) = (1 - 0.3)(\ 0.3) = (0.7)(0.3) = 0.21$
S_3 : R_1 and R_3 occur. $P(S_3) = (0.3)(\ 0.3) = 0.09$

R_1 has two possible outcomes: the cost is zero if the risk does not occur and if the risk occurs, the cost is modeled by a triangular distribution $T(\$8M, \$10M, \$13M)$. R_3 also has two possible outcomes: the cost is zero if the risk does not occur and if the risk occurs, the cost is modeled by a triangular distribution $T(\$13M, \$15M, \$20M)$.

Given these four possible outcomes, we have these states:

$S_0 : P(S_0) = 0.49$, $X_{S_0} = \$0$
$S_1 : P(S_1) = 0.21$; $X_{S_1} = T(\$8M, \$10M, \$13M)$
$S_2 : P(S_2) = 0.21$; $X_{S_2} = T(\$13M, \$15M, \$20M)$
$S_3 : P(S_3) = 0.09$; $X_{S_3} = T(\$8M, \$10M, \$13M) + T(\$13M, \$15M, \$20M)$

The continuous distribution to which we combine these discrete risk states is composed of WBS Elements 1.1 to 1.7. The resulting moments of the continuous distribution (X_{Cont}) are:

$\mu_{X_{Cont}}$ = \$152,860,068.75, and $\sigma_{X_{Cont}}$ = \$4,272,695.15

The statistics of the discrete-risk states ($\mu_{X_{Disc}}$and $\sigma_{X_{Disc}}$) are computed using the calculations of the moments of the triangular distributions and (in the case of S_3, which is a sum of triangular distributions) statistically summing them using Equations 4-10 and 4-11. The distributions of the triangular PDFs of the two risks are uncorrelated, so the standard deviation of the impact of state S_3 is the square root of the sum of the squares of the standard deviations of the two triangular PDFs. The results are

$S_0 : P(S_0) = 0.49$, $\mu_{X_{Disc}}$ = \$0, $\sigma_{X_{Disc}}$ = \$0
$S_1 : P(S_1) = 0.21$; $\mu_{X_{Disc}}$ = \$10,333,333.33 , $\sigma_{X_{Disc}}$ = \$1,027,402.33
$S_2 : P(S_2) = 0.21$; $\mu_{X_{Disc}}$ = \$16,000,000.00 , $\sigma_{X_{Disc}}$ = \$1,471,960.14
$S_3 : P(S_3) = 0.09$; $\mu_{X_{Disc}}$ = \$26,333,333.33 , $\sigma_{X_{Disc}}$ = \$1,795,054.94

To create the mixed distribution of the project cost, $f_{X_m}(x)$, we combine the continuous and discrete distributions using Equation 11-4. $f_{X_m}(x)$ represents the probability-of-occurrence-weighted sum of the PDFs of the individual states.

$$f_{X_m}(x) = \sum_{i=0}^{3} p_{S_i} f_{X_{S_i}}(x) \text{ , where} \qquad 11\text{-}4$$

p_{S_i} = the probability of occurrence of state S_i

$f_{X_{S_i}}(x)$ = the PDF of state S_i

The probabilities of occurrence and statistics used in this operation are shown in Table 11-33.

Table 11-33 Mixed Distribution of Cost Statistics

State	Risk Occurrence	Prob.	μ_X	σ_X	P_X	Q_X
S_0	$\overline{R_1} \cap \overline{R_3}$	0.49	\$152,860,068.75	\$4,272,695.15	18.8446	0.0290
S_1	$R_1 \cap \overline{R_3}$	0.21	\$163,193,402.08	\$4,394,482.83	18.9101	0.0269
S_2	$\overline{R_1} \cap R_3$	0.21	\$168,860,068.75	\$4,519,136.03	18.9442	0.0268
S_3	$R_1 \cap R_3$	0.09	\$179,193,402.08	\$4,634,452.08	19.0036	0.0259

The mixed distribution shown in Figure 11-13 is a plot of $f_{X_m}(x)$. This is a multimodal PDF, and evidence of the discrete components are visible near the means of each state.

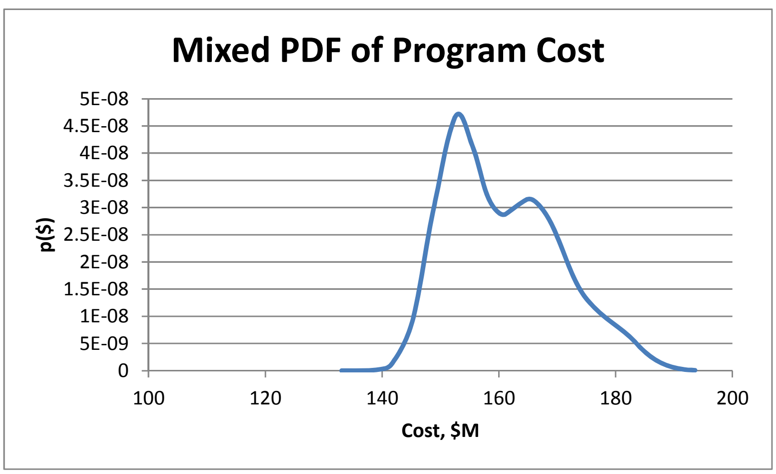

Figure 11-13 Mixed Distribution of Total Cost

When we compare plots of the lognormal approximation to the mixed distribution we see the lognormal approximation captures the overall mean and standard deviation, but it does not accurately portray the multimodal nature of the mixed distribution.

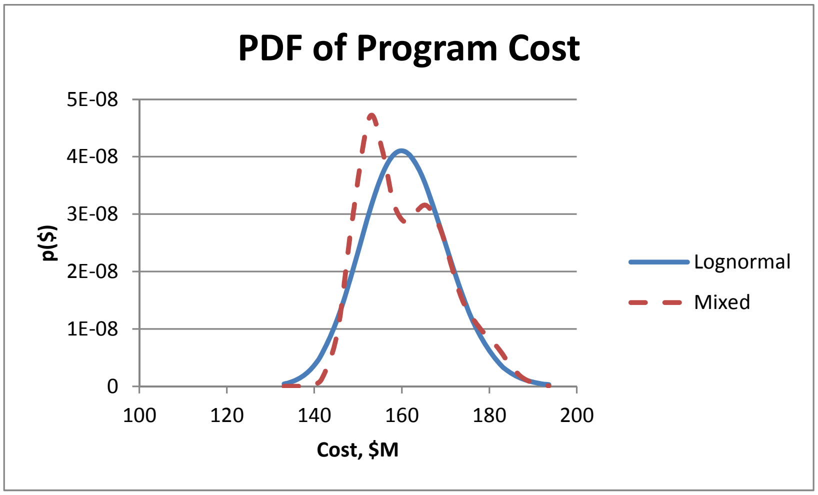

Figure 11-14 Mixed Distribution and Lognormal Approximation of Total Cost

11.2.2.5 Comparison of Total Cost Results

The statistics of the total cost (and their differences) computed using MOM and a 100,000-trial statistical simulation are provided in Table 11-34.

Table 11-34 Total Cost Results from Analytic Approach and Statistical Simulation

	Computed Values		Difference	
	Analytic	Statistical Simulation	Additive	Percent
μ_X	$160,810,256.90	$160,759,226.85	-$51,030.05)	-0.032%
σ_X	$9,765,611.10	$10,064,871.60	-$299,260.50)	-3.064%

11.2.3 Joint Cost and Schedule Distribution

The joint cost and schedule distribution is modeled using a bivariate normal distribution as shown in Equation **11-5**.

$$BiL\left((P_1,P_2),(Q_1,Q_2,\rho_{1,2})\right) = f_{X,D'}(x,d) = \frac{1}{2\pi Q_1 Q_2\sqrt{1-\rho_{1,2}^2 x_1 x_2}} e^{-\left\{\frac{1}{2}w\right\}}; \quad \textbf{11-5}$$

where $w = \frac{1}{1-\rho_{1,2}^2}\left[\left(\frac{ln(x)-P_1}{Q_1}\right)^2 - 2\rho_{1,2}\left(\frac{ln(x)-P_1}{Q_1}\right)\left(\frac{ln(d)-P_2}{Q_2}\right) + \left(\frac{ln(d)-P_2}{Q_2}\right)^2\right]$,

$\rho_{1,2} = \frac{1}{Q_1 Q_2}\ln\left[1+\rho_{X_1,X_2}\sqrt{e^{Q_1{}^2}-1}\sqrt{e^{Q_2{}^2}-1}\right]$, and

ρ_{X_1,X_2} is the correlation coefficient between RVs X_1 and X_2.

The parameters of the lognormal marginal distributions are

$\mu_X = \$160{,}810{,}194.69$, and $\sigma_X = \$11{,}333{,}411.24$

$\mu_{D'} = 872.88wd$ and $\sigma_{D'} = 23.09wd$

The correlation between the total cost and schedule PDFs is calculated using Equation **11-6**.

$\rho_{X,D} = \frac{E[XD]-E[X]E[D]}{\sigma_X \sigma_{D'}} = \frac{E[XD]-\mu_X \mu_{D'}}{\sigma_X \sigma_{D'}}$; where **11-6**
$X = \sum_{i=LLWBS} X_i$, the sum of the costs of the lowest-level WBS elements, X_i
$D = \sum_{j=SSE} D_j$, the sum of the serialized schedule elements (SSE), D_j

It is important to note that there will actually be several correlation coefficients between the cost and schedule PDFs, since each state will have a different set of values in Equation **11-6**. For purposes of this example, we will use the correlation of the combined states.

The sum of the serialized schedule element durations for the NASA example is:

$D = D_{11} + D_{12} + D_{14} + D_{[15,19]} + D_{20} + D_{[23,28]} + D_{29} + D_{30} + D_{32}$, where

$D_{[15,19]} = \max(D_{15}, D_{16} + D_{35} + D_{17}, D_{16} + D_{35} + D_{18}, D_{16} + D_{35} + D_{19})$, and

$D_{[23,28]} = max[max(D_{23}, D_{24}) + D_{27}, D_{25} + D_{36}]$

If we eliminate tasks with $CI = 0$ from D, the serialized schedule equation becomes Equation 11-7.

$D = D_{11} + D_{12} + D_{14} + D_{16} + D_{35} + D_{18} + D_{20} + D_{[23,28]} + D_{29} +$ **11-7**
$D_{30} + D_{32}$, where $D_{[23,28]} = max[max(D_{23}, D_{24}) + D_{27}, D_{25} + D_{36}]$

The product of cost (X) and duration (D), which is a term required to calculate the correlation between them, is the rather large polynomial expression formed by:

$XD = (\sum_{i=EL} X_i)(D_{11} + D_{12} + D_{14} + D_{16} + D_{35} + D_{18} + D_{20} + D_{[23,28]} + D_{29} + D_{30} + D_{32})$

Since the numerator of the correlation equation, $E[XD] - E[X]E[D]$, represents the covariance terms, we only need to account for the correlated durations. X and D are only correlated to each other through their durations, since rates and uncertainties are uncorrelated within the same WBS element. We know the expectation of a squared duration is $E[D_j D_j] = E[{D_j}^2] = {\mu_{D_j}}^2 + {\sigma_{D_j}}^2$

Its contribution to the numerator in the correlation equation will be:

$E[{D_j}^2] - E^2[D_j] = {\sigma_{D_j}}^2$.

This means that any individual task, j, on the critical path with $CI_j = 1$ will have

$\rho_{D_j,D_j} = 1$, and if it is uncorrelated to other tasks, its contribution to the numerator of the correlation equation will be represented simply by ${\sigma_{D_j}}^2$. We also know the expectation of two correlated durations is:

$$E\left[D_k D_j\right] = \mu_{D_k}\mu_{D_j} + \rho_{D_k,D_j}\sigma_{D_k}\sigma_{D_j}$$

Their contribution to the numerator in the correlation equation will be:

$$E\left[D_k D_j\right] - E[D_k]E\left[D_j\right] = \mu_{D_k}\mu_{D_j} + \rho_{D_k,D_j}\sigma_{D_k}\sigma_{D_j} - \mu_{D_k}\mu_{D_j} = \rho_{D_k,D_j}\sigma_{D_k}\sigma_{D_j}$$

The elements of the product XD that will remain in the numerator of the correlation equation are:

1) $R_i\left(\sigma_{D'_i}\right)^2\mu_{\varepsilon_{TD_i}}\mu_{\varepsilon_{TI_i}}$, for tasks $i = [11, 12,14,16,20,29,30,32]$,
2) $R_i\sigma_{D'_i}\mu_{\varepsilon_{TD_i}}\mu_{\varepsilon_{TI_i}}\left(\rho_{D'_i,D'_j}\sigma_{D'_j}\right)$, for tasks $i = [7, 23,24,25,27]$, and $j = \left[7, [23,28]\right]$, and $R_i\sigma_{D'_i}\mu_{\varepsilon_{TD_i}}\mu_{\varepsilon_{TI_i}}$ is substituted with $P(R_3)\sigma_{R_3}$ for task 36.[65]

The first term is quite simple to calculate and results in: 23,476,686.51.

The second term is calculated through the matrix multiplication of the matrix of correlation coefficients between i and j shown in Figure 11-15.

ρ	D7	D11	D12	D16	D35	D18	D20	D[23,28]	D29	D30	D32
7	1	0.1350	0.0675	0.1620	0.4469	0.4321	0.2430	0.5613	0.2670	0.3197	0.1299
11	0.1350	1	0	0	0	0	0	0	0	0	0
12	0.0675	0	1	0	0	0	0	0	0	0	0
16	0.1620	0	0	1	0	0	0	0	0	0	0
35	0.4469	0	0	0	1	0	0	0	0	0	0
18	0.4321	0	0	0	0	1	0	0	0	0	0
20	0.2430	0	0	0	0	0	1	0	0	0	0
23	0.4500	0	0	0	0	0	0	0.2661	0	0	0
24	0.4500	0	0	0	0	0	0	0.2661	0	0	0
25	0.3248	0	0	0	0	0	0	0.3865	0	0	0
27	0.2598	0	0	0	0	0	0	0.1536	0	0	0
29	0.2670	0	0	0	0	0	0	0	1	0	0
30	0.3197	0	0	0	0	0	0	0	0	1	0
32	0.1299	0	0	0	0	0	0	0	0	0	1
36	0.3248	0	0	0	0	0	0	0.6168	0	0	0

Figure 11-15 Matrix of Correlation Coefficients between WBS Elements *i* and Tasks *j*

[65] Task 35 does not have a cost impact, so it does not appear in the term of the product moment.

The resulting calculations give us the numerator of the correlation between cost and schedule, which is 120,005,239.09.

We will use the duration statistics (in wd) to calculate the correlation between cost and schedule. When we use them in the correlation equation, we get:

$$\rho_{X,D} = \frac{E[XD] - E[X]E[D]}{\sigma_X \sigma_{D\prime}} = \frac{120{,}005{,}239.09}{(9{,}765{,}611.10\,)(23.09)} = \; 0.5322$$

The resulting calculations show, for the combined risk states, $\rho_{X,D} = 0.5322$. The results from the 100,000-trial statistical simulation show $\rho_{X,D} = 0.5597$, which is very similar.

Using Equation **11-5**, we are able to provide a three-dimensional plot of the bivariate lognormal PDF of cost and schedule using:

$\mu_X = \$160{,}810{,}256.90$, $\sigma_X = \$9{,}765{,}611.10$

$\mu_{D\prime} = 872.88$ wd, $\sigma_{D\prime} = 23.09$ wd.

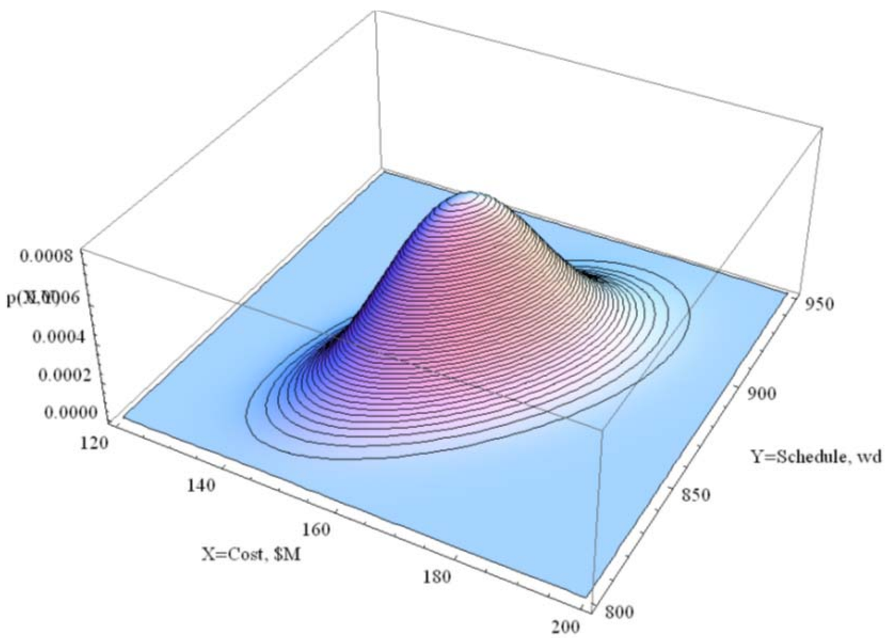

Figure 11-16 Bivariate Lognormal Probability Density of Cost and Schedule

The mixed distribution of cost and schedule relies on the distributions of the individual states, S_i, whose parameters are provided in Table 11-35. The state in which no risks occur, S_0 , accounts for 41.65% of the outcomes. This state has cost and schedule means of $152,860,068.75 and 859.90 *wd*, respectively. The other states have appreciably lower probabilities of occurrence, but their means represent larger values.

Table 11-35 Lognormal Distribution Parameters of Joint Cost and Schedule Probability States

S_i	Risk Occurrence	$P(S_i)$	μ_X	σ_X	$\mu_{D'}$	$\sigma_{D'}$
S_0	$\overline{R_1} \cap \overline{R_2} \cap \overline{R_3}$	0.4165	$152,860,068.75	$4,272,695.15	859.90	19.33
S_1	$\overline{R_1} \cap \overline{R_2} \cap R_3$	0.1785	$168,860,068.75	$4,519,136.03	882.00	19.44
S_{2a}	$\overline{R_1} \cap R_{2a} \cap \overline{R_3}$	0.0245	$152,860,068.75	$4,272,695.15	874.90	19.33
S_{2b}	$\overline{R_1} \cap R_{2b} \cap \overline{R_3}$	0.0245	$152,860,068.75	$4,272,695.15	884.90	19.33
S_{2c}	$\overline{R_1} \cap R_{2b} \cap \overline{R_3}$	0.0245	$152,860,068.75	$4,272,695.15	899.90	19.33
S_{3a}	$\overline{R_1} \cap R_{2a} \cap R_3$	0.0105	$168,860,068.75	$4,519,136.03	897.00	19.44
S_{3b}	$\overline{R_1} \cap R_{2b} \cap R_3$	0.0105	$168,860,068.75	$4,519,136.03	907.00	19.44
S_{3c}	$\overline{R_1} \cap R_{2c} \cap R_3$	0.0105	$168,860,068.75	$4,519,136.03	922.00	19.44
S_4	$R_1 \cap \overline{R_2} \cap \overline{R_3}$	0.1785	$163,193,402.08	$4,394,482.83	859.90	19.33
S_5	$R_1 \cap \overline{R_2} \cap R_3$	0.0765	$179,193,402.08	$4,634,452.08	882.00	19.44
S_{6a}	$R_1 \cap R_{2a} \cap \overline{R_3}$	0.0105	$163,193,402.08	$4,394,482.83	874.90	19.33
S_{6b}	$R_1 \cap R_{2b} \cap \overline{R_3}$	0.0105	$163,193,402.08	$4,394,482.83	884.90	19.33
S_{6c}	$R_1 \cap R_{2c} \cap \overline{R_3}$	0.0105	$163,193,402.08	$4,394,482.83	899.90	19.33
S_{7a}	$R_1 \cap R_{2a} \cap R_3$	0.0045	$179,193,402.08	$4,634,452.08	897.00	19.44
S_{7b}	$R_1 \cap R_{2b} \cap R_3$	0.0045	$179,193,402.08	$4,634,452.08	907.00	19.44
S_{7c}	$R_1 \cap R_{2c} \cap R_3$	0.0045	$179,193,402.08	$4,634,452.08	922.00	19.44

The joint PDF formed is a mixture distribution formed by the probability-weighted joint PDFs of each state (Figure 11-17). Note the variance of the mixed distribution is much greater than that of any of the individual states. This is due to the variance contribution of each state's distance to the mean of the mixed distribution.

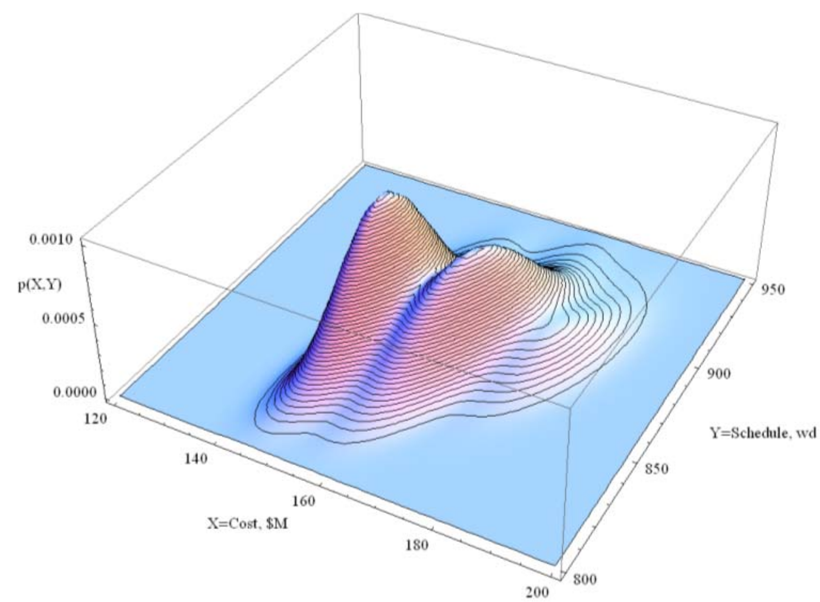

Figure 11-17 Joint Probability Density of Cost and Schedule

The probability that the project will cost equal to or less than the point estimate of cost (PEX) and will be completed on or before the schedule point estimate (PED) is evaluated through integration of the following joint cost and schedule PDF:

$$P[d \leq PED; x \leq PEX] = \int_0^{PEX} \int_0^{\mathrm{PED}} f_{D,X}(d,x)\, dDdx$$

We can set the upper limits of the integral above using the point estimate for cost, $PEX = \$151{,}500{,}000$, and the point estimate for schedule, $PED = 1173cd$ or $840wd$.

Since the distribution $f_{D,X}(d,x)$ is a mixture distribution with 16 possible states, we can express the joint probability as the probability-weighted sum:

$$P[d \leq PED; x \leq PEX] = \sum_{i=0}^{7c} P(S_i) \int_0^{PEX} \int_0^{\mathrm{PED}} f_{D,X_{S_i}}(d,x)\, dDdx = \sum_{i=0}^{7c} P(S_i) J_{S_i}$$

This results in the set of sixteen joint probabilities ($P(S_i)$) and probability-weighted joint probabilities ($P(S_i)J_{S_i}$), as shown in Table 11-36. The sum of $P(S_i)J_{S_i}$, which represents the joint probability of the point estimates of cost and schedule, is 0.04766, or 4.766%, which is extremely low. $P(S_0)J_{S_0}$ is 4.630%, which accounts for nearly all of the joint probability. This is because state S_0 has the highest joint probability density at the x, d coordinates of the point estimates of cost and schedule duration. The marginal cost and schedule variances of all of the states are similar; however the means of the risk-included states are all higher than that of S_0.

Table 11-36 Joint Probabilities of Possible Risk States

S_i	Risk Occurrence	$P(S_i)$	J_{S_i}	$P(S_i)J_{S_i}$
S_0	$\overline{R_1} \cap \overline{R_2} \cap \overline{R_3}$	0.4165	1.11E-01	0.046304804
S_1	$\overline{R_1} \cap \overline{R_2} \cap R_3$	0.1785	1.37E-05	2.45409E-06
S_{2a}	$\overline{R_1} \cap R_{2a} \cap \overline{R_3}$	0.0245	2.87E-02	0.000704228
S_{2b}	$\overline{R_1} \cap R_{2b} \cap \overline{R_3}$	0.0245	8.09E-03	0.000198291
S_{2c}	$\overline{R_1} \cap R_{2b} \cap \overline{R_3}$	0.0245	6.75E-04	1.65497E-05
S_{3a}	$\overline{R_1} \cap R_{2a} \cap R_3$	0.0105	4.41E-06	4.62865E-08
S_{3b}	$\overline{R_1} \cap R_{2b} \cap R_3$	0.0105	1.30E-06	1.36871E-08
S_{3c}	$\overline{R_1} \cap R_{2c} \cap R_3$	0.0105	9.49E-08	9.9621E-10
S_4	$R_1 \cap \overline{R_2} \cap \overline{R_3}$	0.1785	2.33E-03	0.000415305
S_5	$R_1 \cap \overline{R_2} \cap R_3$	0.0765	4.34E-11	3.31634E-12
S_{6a}	$R_1 \cap R_{2a} \cap \overline{R_3}$	0.0105	1.28E-03	1.34409E-05
S_{6b}	$R_1 \cap R_{2b} \cap \overline{R_3}$	0.0105	6.18E-04	6.48392E-06
S_{6c}	$R_1 \cap R_{2c} \cap \overline{R_3}$	0.0105	1.11E-04	1.1689E-06
S_{7a}	$R_1 \cap R_{2a} \cap R_3$	0.0045	3.29E-11	1.48011E-13
S_{7b}	$R_1 \cap R_{2b} \cap R_3$	0.0045	2.08E-11	9.37364E-14
S_{7c}	$R_1 \cap R_{2c} \cap R_3$	0.0045	5.63E-12	2.53355E-14
Total		1.0000		0.04766

12 Summary

This report presents an analytic (i.e., a non-simulation based) method of quantitative cost and schedule risk analysis building on analytic techniques of applied probability and statistics. The analytic method provides near-instantaneous results with exact statistics such as mean and variance of total cost and total schedule duration. It capitalizes on the fact that the structures of both cost and schedule estimates define mathematical problems to be solved through the use of applied probability. In this report we provide the mathematics required to perform the task of 1) calculating the uncertainty of an estimate, 2) determining the risk from this uncertainty and a point estimate.

While much of the mathematics of applied probability used in this report are publicly available through journal publications, the authors have derived methods and formulae for functional correlation and application of discrete risks that have never been published before. Therefore the report provides a very unique set of mathematics useful in the analytic assessment of cost and schedule uncertainty and risk.

The report includes several quantitative examples, including two example estimates, where the results obtained using the analytic method compare well with those results obtained through statistical simulation. In cases where large-tailed distributions were involved in the analysis (e.g., when discrete risks are used in an estimate or when we wish to find the product of two or more RVs) we found simulations require very large number of trials and often did not provide correct or even stable answers from run to run.

Given the excellent results obtained through this research, additional applications of the analytic method are recommended for use in risk analysis, estimating relationship development and probabilistic cost and schedule estimating.

13 Conclusions and Recommendations

13.1 Conclusions

In the course of this research, perhaps the most daunting task was how to perform analytic cost risk analysis using analogies and cost-on-cost factors. On the surface, these cost estimating methods are simple and easy to understand, but they have much larger, more complicated, and perhaps even sinister implications when treating them probabilistically.

The first issue is how to model probability distribution of an analogy, which is discussed in Section 3.2.2.2. Without specifying the analogy as the mean or as a particular percentile of the PDF, the distribution parameters are difficult to calculate. As pointed-out in the literature (Flynn, Braxton, Garvey, & Lee, 2012), specifying a percentile value for an analogy reduces the problem enormously.

The second issue is the difficulty in proper derivation and use of the cost-dependent CER or factor. Anderson and Covert (Reducing Systemic Errors in Cost Models, 2008), (Regression of Cost Dependent CERs, 2002) discuss how to properly develop these factors – which is correct, but not the current industry norm. Additionally, the use of cost-dependent CERs in a probabilistic uncertainty analysis requires the calculation of the statistics of the product of the individual uncertainties. Calculating the moments of the product of two lognormal distributions is a difficult task to perform analytically and is particularly difficult for statistical simulations to do correctly and consistently from one simulation run to another. The analyst understanding the probabilistic implications of using cost-dependent CERs in an estimate will gain a healthy respect for these functions bordering on a strong dislike of them.

The final conclusions we draw from this research are that analytic methods provide exact, near-instantaneous results in cost and schedule (and joint cost and schedule) uncertainty analysis. The mathematics used in the analysis require a significant non-recurring set-up time and are best suited for models that have a defined WBS, such as the NASA/Air Force Cost Model (NAFCOM), the NASA Instrument Cost Model (NICM) and the Unmanned Space Vehicle Cost Model (USCM). The methods provided in this report would be a great improvement to the performance of the risk analysis capabilities of these models.

13.2 Recommendations

The following set of recommendations provides avenues for continuing research in the area of applied probability with applications to probabilistic cost and schedule risk analysis. This research will improve the understanding of cost and schedule estimating through the application of uncertainty in our estimates, which are uncertain predictions of future events.

13.2.1 Evaluating Statistical Simulations

This research provides many examples whereby the exact statistics of RVs and functions of dependent RVs are compared to the results from statistical simulations. In some cases, particularly when computing the product of two lognormal random variables and when discrete risks are included in an estimate, the results of the statistical simulation are not close enough approximations to ignore simulation error. The ability to extract statistical data from simulations is important because it allows the analyst to determine how the simulation arrived at a particular set of results. We recommend developing a small set of test cases and experiments to determine the quality of statistical simulation tools that can be compared to the exact values computed with the equations and methods presented in this report.

13.2.2 Using Estimating Methods

The results of this research have indicated that estimates relying on methods such as build-up approaches and direct analogies may require additional cost and schedule risks to be included in them. Estimates using multiple scaled actuals or CERs that are created from a database of actual costs and schedule durations from similar programs require fewer risks to be included, presumably because the actual costs and schedule durations in the database will include risks that have occurred. We recommend performing a study that compares the risk-estimating ability of different estimating methods to determine whether or not using estimating methods derived from multiple scaled actuals is a better predictor of estimating uncertainty.

13.2.3 Basis of Estimate Credibility

Basis of estimate (BOE) credibility can be enhanced by use of multiple scaled actuals / CERs as either a primary or secondary estimating method. BOEs based on expert judgment and analogies require inclusion of discrete risks to account for missing risks in the estimate. Discrete risk formulations such as those described in Section 9 provide a method of accounting for discrete risks and the uncertainty due to them. BOEs based on CERs or multiple scaled actuals require fewer discrete risks to be applied to the estimate and provide a more substantive estimate.

13.2.4 Developing Cost Models

CER regression techniques have traditionally been limited to curve fitting of vectors of discrete dependent variables (cost) with vectors of discrete independent variables (cost drivers) as shown in Figure 13-1.

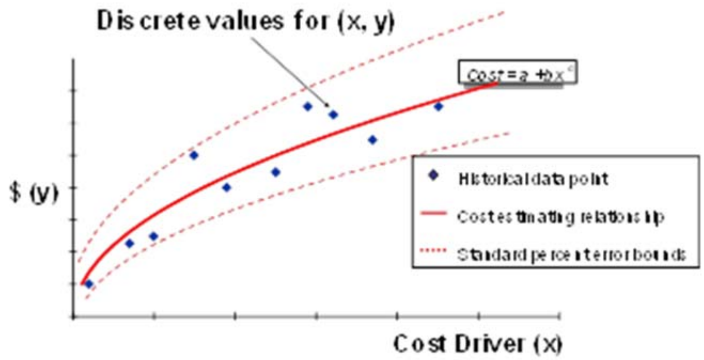

Figure 13-1 Regression of Discrete Variables

We assume the regression variables are discrete and non-random in nature; however, errors in both the dependent and independent variables can arise in the data collection and normalization process (Figure 13-2). Error-in-variables (EIV) regression techniques can be employed to find appropriate CERs with errors in either the dependent or independent variables or even when both are random variables (Covert R. P., 2006).[66] Using the analytic method in the CER development process makes a non-simulation-based EIV regression technique feasible and allows the CER developer to instantaneously see the true error effects of CER regressions on cost model errors.

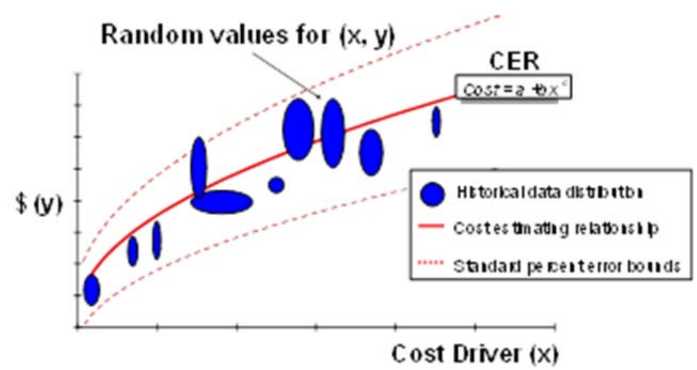

Figure 13-2 Regression of Random Variables

13.2.5 Improving Cost and Schedule Risk Tools

Cost models such as NAFCOM, NICM, USCM and the Aerospace Small Satellite Cost Model (SSCM) are all good candidates for implementing the analytic methods of uncertainty analysis shown in this report.

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[66] Covert, R., "Errors-In-Variables Regression", Joint SSCAG/EACE/SCAF Meeting, London, UK, September 19-21, 2006.

Currently, NAFCOM uses a two-step process to model cost risk since the methods and equations for calculating functional correlation were unknown at the time NAFCOM implemented its cost risk analysis method based on FRISK. In the first step, uncertainty is calculated for the prime mission product (PMP). In the second step, uncertainty for cost-on-cost functions such as System Engineering, Integration Assembly & Test is calculated. We recommend replacing the method of CRA in NAFCOM to instantaneously calculate exact means and variances of total cost distributions in a single-step approach using the methods proposed in this report rather than through a two-step approach. This will provide exact answers and increase computational efficiency.

13.2.6 Time-Phasing a Resource-Loaded Schedule

A natural extension of the second example problem in this report is to include time-phasing of a resource-loaded cost and schedule estimate. Using what we have learned about using probability distributions of cost and schedule duration (i.e., uniform, triangular, beta), we can apply the same principles to distributions of resources over time. The resulting information that could be obtained from a time-phased, resource-loaded schedule estimate will be a multivariate distribution of probability with respect to cost, schedule and time. Combining these in a probabilistic estimate would allow the analyst to compute joint probability/resource-loaded cost and schedule estimates. Conditional values of cost and schedule duration would be easily obtained as well as the joint probability distribution.

13.2.7 Allocating Schedule Margin

Allocating margins to schedule tasks (or groups of tasks) is important to ensure projects do not overrun their schedules. Several methods have been proposed that use the results of statistical simulations to reverse-and-forward-allocate schedule margin. These methods start with a confidence level of the probabilistic finish date and back-allocate schedule reserve to tasks along the critical path to the starting task. Then the schedules with reserve are recalculated to compute the new point estimate of the finish date.

Book (2006) proposed a method of cost risk allocation based on the "needs" of particular WBS elements required to achieve a particular confidence level.[67] This method has not been applied to schedule estimating prior to this report, to our knowledge, since the effective linearization of the schedule network problem has not been widely published. We believe that "linearized" schedule networks such as the one demonstrated in Section 11.2 provide the necessary mathematical structure to allow schedule allocation based on need. We propose developing a risk allocation method using these principles.

[67] Book, S. A. (2006). Allocating Risk Dollars Back to WBS Elements. ISPA/SCEA Joint Conference and Training Workshop. Seattle, WA.

14 Acronyms, Symbols and Definitions

14.1 Acronyms

AIAA	American Institute of Aeronautics and Astronautics
ADACS	Attitude determination and control system
AGE	Aerospace ground equipment
ATP	Authority to proceed
BOE	Basis of Estimate
BOLP	Beginning-of-life power
cd	Calendar days
CDF	Cumulative distribution function
CDF^{-1}	Inverse cumulative distribution function
C&DH	Command and data handling
CDR	Critical Design Review
CER	Cost estimating relationship
CI	Criticality index
CMF	Cumulative mass function
CRA	Cost risk analysis
CTV	Contribution to variance
EIV	Errors-in-Variables
FGM	Farlie-Gumbel-Morgenstern
FRISK	Formal Risk Assessment of System Cost
GFLOPs	Giga (billions of) floating point operations per second
IA&T	Integration, assembly and test
IEEE	Institute of Electrical and Electronics Engineers
iff	If and only if
JACS	Joint Analysis of Cost and Schedule
JCS	Joint cost and schedule
LLWBS	Lowest-level work breakdown structure [element]
LOOS	Launch and orbital operations support
MOM	Method of moments
NASA	National Aeronautics and Space Administration
NAFCOM	NASA/Air Force Cost Model
PDF	Probability density function
NICM	NASA Instrument Cost Model
PDR	Preliminary Design Review
PM	Project management
PMF	Probability mass function
PMP	Prime mission product
ROR	Risk and opportunities register
RV	Random variable
SEITPM	Systems engineering, integration and test, and program management
SOS	System-of-Systems
TCS	Thermal control system

TD	Time-dependent
TI	Time-independent
TTC	Telemetry, tracking and command/control
USCM	Unmanned Space Vehicle Cost Model
UV	Ultraviolet
WBS	Work breakdown structure
wd	Workdays
WRT	With respect to

14.2 Symbols

a, b, c, d	Coefficients a through d
e	Naperian base
ε_i	Error i
f, g, h	Functions
i, j, k, l	Indices i through l
m, n	Counters
p	The probability a particular event occurs
D_j	Risk impact j
P, Q	Lognormal shape parameters
$E[X]$	Expectation of X
$\delta\mu$	Difference of two means
$\delta\sigma$	Difference of two standard deviations
D'	Duration in workdays
F'	Finish date in consecutive calendar days
$f_X(x), g_X(x)$	PDFs of f_X and g_X over x
$F_X(x), G_X(x)$	CDFs of f_X and g_X
$Max(X, Y)$	Maximum of X and Y
$Var(X)$	Variance of X
$Corr(X, Y)$	Pearson correlation of RVs X and Y
$Cov(X, Y)$	Covariance of X and Y
$\rho_{X,j}$	Pearson correlation of RVs X and Y
σ_{XY}	Covariance of X and Y
μ	Mean of X
μ'_k	k^{th} Raw moment of X
σ_X	Standard deviation of X
${\sigma_X}^2$	Variance of X
$U(L, H)$	Uniform distribution defined by L and H
$T(L, M, H)$	Triangular distribution defined by L, M and H
$N(\mu, \sigma)$	Normal distribution defined by μ and σ
$L(P, Q)$	Lognormal distribution defined by P and Q
$B(\alpha, \beta, L, H)$	Beta distribution defined by shape parameters α, β, and limits L, H

$\gamma_{i,j}$	Binary value of bit j of integer i
R_i	Risk i
$\rho_{X,Y}$	Linear (Pearson) correlation coefficient for X and Y
ρ_{X,Y_S}	Rank (Spearman) correlation coefficient for X and Y
S_i	[Risk] state i
φ	PDF of Standard Normal Distribution
Φ	CDF of Standard Normal Distribution
v	Skewness
κ	Kurtosis
$\cap$	Boolean "and"
$\bar{R}_1$	Boolean "not" of risk i

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16 Appendices

16.1 Appendix A – Probability Distributions

16.1.1 Uniform Distribution

The uniform distribution is defined by two parameters: The minimum possible value (*L*), and the maximum possible value (*H*).

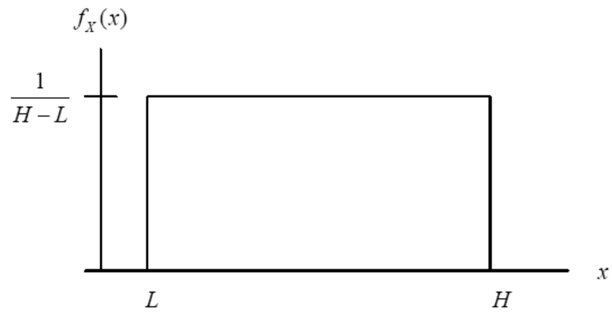

Figure 16-1 Uniform Distribution

The PDF of the uniform distribution $U(L, H)$ is:

$$f_X(x) = \frac{1}{(H-L)} \quad , if\ L \leq x \leq H \qquad \textbf{16-1}$$

The CDF of the uniform distribution $U(L, H)$ is:

$$F_X(x) = \begin{cases} 0 & , if\ x < L \\ \frac{(x-L)}{(H-L)} & , if\ L \leq x \leq H \\ 1 & , if\ x > H \end{cases} \qquad \textbf{16-2}$$

Its mean, or expected value, E(X), is:

$$E(X) = \frac{L+H}{2} \qquad \textbf{16-3}$$

And its variance, *Var(X)*, is:

$$Var(X) = \frac{1}{12}(H-L)^2 \qquad \textbf{16-4}$$

Higher order moments such as skewness and kurtosis are:

$$Skew(X) = 0 \qquad \textbf{16-5}$$
$$Kurt(X) = -6/5 \qquad \textbf{16-6}$$

16.1.2 Triangular Distribution

The triangular distribution is defined by three parameters, the lowest possible value (L), the mode (M), and the highest possible value (H).

The PDF of the triangular distribution $T(L, M, H)$ is:

$$f_X(x) = \begin{cases} \frac{2(x-L)}{(H-L)(M-L)} & if\ L \le x < M \\ \frac{2(H-x)}{(H-L)(H-M)} & if\ M \le x \le H \end{cases} \quad \textbf{16-7}$$

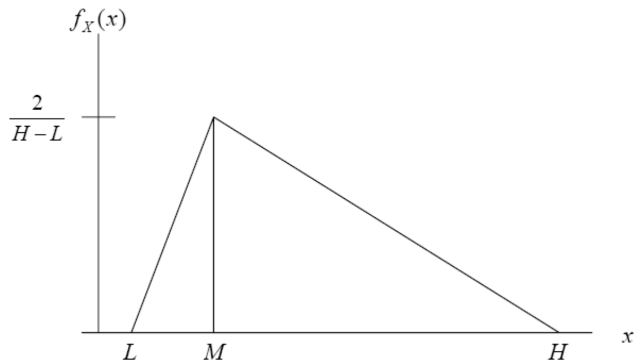

Figure 16-2 Triangular Distribution

If X is a triangular random variable, then its mean, or expected value, *E(X)*, is:

$$E(X) = \frac{(L+M+H)}{3} \quad \textbf{16-8}$$

its variance, *Var(X)*, is:

$$Var(X) = \frac{1}{18}[(M-L)(M-H) + (H-L)^2] \quad \textbf{16-9}$$

Higher order moments such as skewness and kurtosis are:

$$Skew(X) = \frac{\sqrt{2}(L+H-2M)(2L-H-M)(L-2H+M)}{5\sqrt{(L^2+M^2+H^2-LH-LM-MH)^3}} \quad \textbf{16-10}$$

$$Kurt(X) = -3/5 \quad \textbf{16-11}$$

16.1.3 Normal Distribution

The normal PDF is uniquely defined by the parameters μ and σ.

The normal distribution $N(\mu, \sigma)$ is defined by the following PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\left\{\frac{1}{2}\left[\left[\frac{(t-\mu)^2}{\sigma^2}\right]\right]\right\}} \quad \textbf{16-12}$$

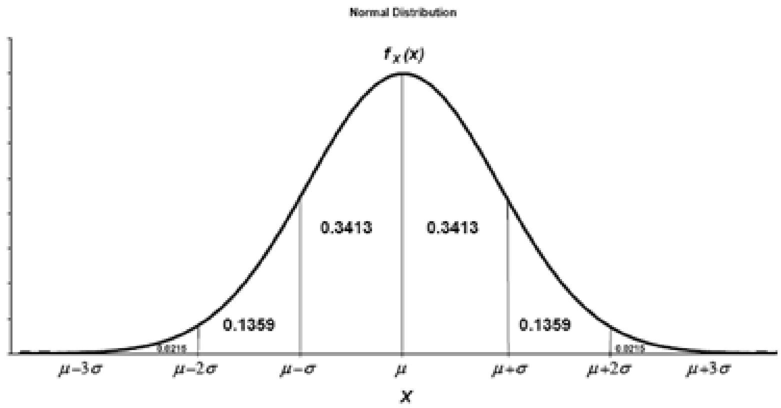

Figure 16-3 Normal Distribution from (Garvey, 2000)

The CDF of the normal distribution is often of interest, since it enables calculation of the percentiles of the distribution. The CDF of the normal distribution is defined as follows:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left\{\frac{1}{2}\left[\left[\frac{(t-\mu)^2}{\sigma^2}\right]\right]\right\}} dt \qquad \textbf{16-13}$$

Higher order moments such as skewness and kurtosis are:

$$Skew(X) = 0 \qquad \textbf{16-14}$$
$$Kurt(X) = 3 \qquad \textbf{16-15}$$

16.1.4 Lognormal Distribution

A lognormal random variable is the exponentiation of a normal random variable. Because the lognormal random variable (*X*) and the normal random variable (*Y*) are related, their means and standard deviations are also related.

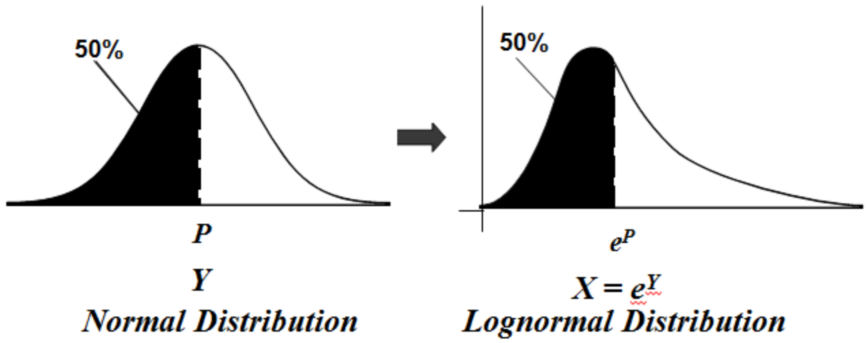

Figure 16-4 Transformation of Lognormal Distribution

Other important statistics associated with the lognormal distribution are the mode and median:

$$Mode(X) = e^{\mu_Y - \sigma_Y{}^2} = e^{P-Q^2} \quad \textbf{16-16}$$

$$Median(X) = e^{\mu_Y} = e^{P} \quad \textbf{16-17}$$

The PDF of the lognormal distribution is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}Qx} e^{-\left\{\frac{1}{2}\left[\left[\frac{(\ln(x)-P)^2}{Q^2}\right]\right]\right\}} \quad \textbf{16-18}$$

and the CDF of the lognormal distribution is:

$$F_X(x) = P(X \le x) = \int_0^x \frac{1}{\sqrt{2\pi}Qt} e^{-\left\{\frac{1}{2}\left[\left[\frac{(\ln(t)-P)^2}{Q^2}\right]\right]\right\}} dt \quad \textbf{16-19}$$

16.1.5 Beta Distribution

16.1.5.1 Standard Beta Distribution

The standard beta distribution, $Beta(\alpha, \beta)$, is defined by two shape parameters, α and β over the interval [0,1].

The PDF of $B(\alpha, \beta)$ is

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad \textbf{16-20}$$

With mean,

$$E[X] = \frac{\alpha}{\alpha+\beta} \quad \textbf{16-21}$$

and variance,

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad \textbf{16-22}$$

The k^th moment of $Bs(\alpha, \beta)$ is

$$E[X^k] = \frac{\alpha+k-1}{\alpha+\beta+k-1} E[X^{k-1}] \quad \textbf{16-23}$$

which is a recursive equation.

16.1.5.2 The Four Parameter Beta Distribution

The four parameter beta distribution, $Beta4(\alpha, \beta, a, b)$, is defined by four parameters: α and β (which are the standard Beta shape parameters); and support parameters a and b (which are the minimum and maximum bounds of the distribution, respectively).

The PDF of $Beta4(\alpha, \beta, a, b)$ is obtained through affine transformation of the standard beta distribution which changes the support from [0,1] to [a,b].

$$f_Y(y) = \frac{1}{(b-a)} f_X\left(\frac{y-a}{b-a}\right) = \frac{1}{(b-a)} \frac{\left(\frac{y-a}{b-a}\right)^{\alpha-1}\left(1-\left(\frac{y-a}{b-a}\right)\right)^{\beta-1}}{B(\alpha,\beta)} \quad \textbf{16-24}$$
$$= \frac{(y-a)^{\alpha-1}(c-y)^{\beta-1}}{(b-a)^{\alpha+\beta-1}B(\alpha,\beta)}; (a < y < b)$$

With mode,

$$m = \left(\frac{\alpha-1}{\alpha+\beta-2}\right) b + \left(\frac{\beta-1}{\alpha+\beta-2}\right) a \quad \textbf{16-25}$$

mean,

$$E[Y] = \left(\frac{\alpha}{\alpha+\beta}\right) b + \left(\frac{\beta}{\alpha+\beta}\right) a \quad \textbf{16-26}$$

and variance,

$$Var(X) = \frac{\alpha\beta(b-a)^2}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad \textbf{16-27}$$

16.1.5.3 The PERT Distribution

The PERT distribution, $P(a, m, b)$, is a special case of the four parameter beta distribution whereby: 1) the parameters a and b are the maximum and minimum bounds of the distribution; 2) the mode, m, is explicitly defined; and 3) the mean and variance obey strict definitions:

Mean, $$\mu = E[X] = \frac{a+4m+b}{6} \qquad \textbf{16-28}$$

Variance, $$Var(X) = \frac{(b-a)^2}{36} \qquad \textbf{16-29}$$

For 16-28 and 16-29 to hold true for the PERT distribution, the standard beta parameters, α and β, are derived from $P(a, m, b)$ by[68]

$$\alpha = \frac{(\mu-a)(2m-a-b)}{(m-\mu)(b-a)} \text{ and} \qquad \textbf{16-30}$$
$$\beta = \frac{\alpha(b-\mu)}{(\mu-a)} \text{ where}$$
$$\mu = \frac{a+4m+b}{6}$$

For the symmetric case, the standard beta parameters α and β must satisfy this condition: [69]

If $m = \frac{b+a}{2}$, then $\alpha = 3$ and $\beta = 3$ (proof of this is provided in Appendix C – Derivations)

16.1.6 Bivariate Normal Distribution

The bivariate normal distribution is a joint distribution formed by two normal distributions and is defined by

$$BiN\left((\mu_1, \mu_2), (\sigma_1, \sigma_2, \rho_{1,2})\right) = f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{1,2}^2}} e^{-\left\{\frac{1}{2}w\right\}}; \qquad \textbf{16-31}$$

where $$w = \frac{1}{1-\rho_{1,2}^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{1,2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right],$$
$$\rho_{1,2} = \rho_{X_1,X_2}$$

16.1.7 Bivariate Normal-Lognormal Distribution

$$BiNL\left((\mu_1, \mu_2), (\sigma_1, \sigma_2, \rho_{1,2})\right) = f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1 Q_2\sqrt{1-\rho_{1,2}^2}\,x_2} e^{-\left\{\frac{1}{2}w\right\}}; \qquad \textbf{16-32}$$

where $$w = \frac{1}{1-\rho_{1,2}^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{1,2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{ln(x_2)-P_2}{Q_2}\right) + \left(\frac{ln(x_2)-P_2}{Q_2}\right)^2\right],$$

[68] From Vose Software ModelRisk Help, © Vose Software™ 2007. Reference Number: M-M0361-A

[69] Note the Beta Distribution article in Wikipedia, as accessed 13 November 2012, does not correctly specify these formulae and states that for the symmetric case that $\alpha = 4$ and $\beta = 4$, which are incorrect.

P_2 is defined by $P = \frac{1}{2} ln\left(\frac{\mu^4}{\mu^2+\sigma^2}\right)$, Q_2 is defined by $Q = \sqrt{ln\left(1+\frac{\sigma^2}{\mu^2}\right)}$, $\rho_{1,2} = \rho_{X_1,X_2} \frac{\sqrt{e^{Q_2^2}-1}}{Q_2}$

16.1.8 Bivariate Lognormal Distribution

$$BiL\left((\mu_1,\mu_2),(\sigma_1,\sigma_2,\rho_{1,2})\right) = f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi Q_1 Q_2\sqrt{1-\rho_{1,2}^2}x_1 x_2} e^{-\left\{\frac{1}{2}w\right\}}; \quad \textbf{16-33}$$

where $w = \frac{1}{1-\rho_{1,2}^2}\left[\left(\frac{ln(x_1)-P_1}{Q_1}\right)^2 - 2\rho_{1,2}\left(\frac{ln(x_1)-P_1}{Q_1}\right)\left(\frac{ln(x_2)-P_2}{Q_2}\right) + \left(\frac{ln(x_2)-P_2}{Q_2}\right)^2\right]$,

P_1 and P_2 are defined by $P = \frac{1}{2} ln\left(\frac{\mu^4}{\mu^2+\sigma^2}\right)$, Q_1 and Q_2 are defined by $Q = \sqrt{ln\left(1+\frac{\sigma^2}{\mu^2}\right)}$, $\rho_{1,2} = \frac{1}{Q_1 Q_2} ln\left(1+\rho_{X_1,X_2}{}^{\sqrt{e^{Q_1^2}-1}\sqrt{e^{Q_2^2}-1}}\right)$

16.2 Appendix B – Expectation Operations

16.2.1 Expectation Properties

If X is a PDF then the expected value of X is:

$$\mathrm{E[X]} = \mu_X \qquad \textbf{16-34}$$

The variance of X is:

$$Var(X) = E[X^2] - E^2[X] = E[X^2] - {\mu_X}^2 \qquad \textbf{16-35}$$

The covariance of X and Y is:

$$Cov(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] \qquad \textbf{16-36}$$
$$Cov(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y \qquad \textbf{16-37}$$
$$Cov(X,Y) = E[XY] - \mu_X\mu_Y \qquad \textbf{16-38}$$
$$Cov(X,Y) = Cov(Y,X) \qquad \textbf{16-39}$$
$$Cov(aX+b, cY+d) = (ac)Cov(X,Y) \qquad \textbf{16-40}$$

If X and Y are independent, then $Cov(X,Y) = 0$ **16-41**

$$E[XY] = \rho_{X,Y}\sigma_X\sigma_Y + \mu_X\mu_Y \qquad \textbf{16-42}$$
$$Corr(X,Y) = \rho_{X,Y} = \frac{E[XY]-\mu_X\mu_Y}{\sigma_X\sigma_Y} \qquad \textbf{16-43}$$

$$E(a+bX) = a + bE(X) = a + b\mu_X \qquad \textbf{16-44}$$
$$Var(a+bX) = (b^2)Var(X) \qquad \textbf{16-45}$$

The k^{th} moment of X

$$E[X^k] = \begin{cases} \sum_X x^k P_X(x) & ,if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^k f_X(x)dx & ,if\ X\ is\ continuous \end{cases} \qquad \textbf{16-46}$$

$$E[g(x)] = \begin{cases} \sum_X g(x) P_X(x) & ,if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} g(x) f_X(x)dx & ,if\ X\ is\ continuous \end{cases} \qquad \textbf{16-47}$$

$$E[X+Y] = E[X] + E[Y] \qquad \textbf{16-48}$$
$$E[X-Y] = E[X] - E[Y] \qquad \textbf{16-49}$$
$$Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y) \qquad \textbf{16-50}$$
$$Var[X-Y] = Var[X] + Var[Y] - 2Cov(X,Y) \qquad \textbf{16-51}$$

16.2.2 Expectation Operations

For the uniform case, where $f_X(x) = \frac{1}{H-L}$

$$E[X^{\mathrm{k}}] = \int_{-\infty}^{\infty} x^{\mathrm{k}} f_X(x)\,dx = \frac{1}{H-L}\int_{-\infty}^{\infty} x^{\mathrm{k}}\,dx = \frac{1}{[H-L]}\frac{1}{[\mathrm{k}+1]}x^{\mathrm{k}+1}\Big|_L^H = \frac{H^{\mathrm{k}+1} - L^{\mathrm{k}+1}}{\mathrm{k}+1(H-L)}$$

For the triangular case

$$\mathrm{E}[\mathrm{X}^{\mathrm{k}}] = \frac{2}{(\mathrm{H}-\mathrm{L})(\mathrm{M}-\mathrm{L})}\left\{\frac{\mathrm{M}^{\mathrm{k}+2}-\mathrm{L}^{\mathrm{k}+2}}{\mathrm{k}+2} - \mathrm{L}\frac{\mathrm{M}^{\mathrm{k}+1}-\mathrm{L}^{\mathrm{k}+1}}{\mathrm{k}+1}\right\} + \frac{2}{(\mathrm{H}-\mathrm{L})(\mathrm{H}-\mathrm{M})}\left\{\mathrm{H}\frac{\mathrm{H}^{\mathrm{k}+1}-\mathrm{M}^{\mathrm{k}+1}}{\mathrm{k}+1} - \frac{\mathrm{H}^{\mathrm{k}+2}-\mathrm{M}^{\mathrm{k}+2}}{\mathrm{k}+2}\right\} \quad \textbf{16-52}$$

For the normal case (by definition), k is defined as a positive integer. In cases where k is not an integer value, $\mathrm{E}[\mathrm{X}^{\mathrm{k}}]$ is defined by a series of confluent hypergeometric equations.

$$E[X^0] = 1$$

$$E[X^1] = \mu$$

$$E[X^2] = \mu^2 + \sigma^2$$

$$E[X^3] = \mu^3 + 3\mu\sigma^2$$

$$E[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

For the lognormal case from Garvey (2000), $E[X^{\mathrm{k}}]$ is defined for all positive values of k.

$$E[X^{\mathrm{k}}] = e^{\left(kP+\frac{1}{2}Q^2k^2\right)}$$