Constructing possibility distribution from a time series

The following content has been taken from the following article.

Ullah, A.M.M.S.; Shamsuzzaman, M. Fuzzy Monte Carlo Simulation using point-cloud-based probability–possibility transformation. *Simulation* **2013**, *89*, 860–875, https://doi.org/10.1177/0037549713482174

Let $x(t) \in \Re$, $t = 0, \ldots, n-1$ be n data points in the form of a time series, as shown in Figure B1.

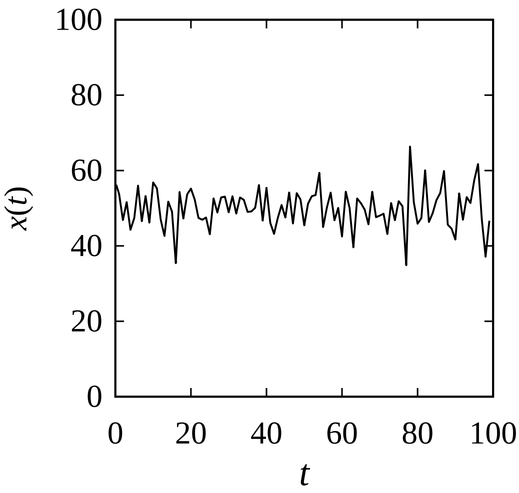

Figure B1. A given set of numerical data.

Let $(x(t), x(t+1))$, $t = 0, \ldots, n-1$ be a point-cloud in the universe of discourse $X = [x_{min}, x_{max}]$ so that $x_{min} < \min(x(t) \mid \forall t \in \{0, \ldots, n\})$ and $x_{max} > \max(x(t) \mid \forall t \in \{0, \ldots, n\})$. Let A and B two square boundaries so that the vectors of the vertices of A and B (in the anti-clockwise direction) are $((x_{min}, x_{min}), (x, x_{min}), (x, x), (x_{min}, x))$ and $((x_{max}, x_{max}), (x, x_{max}), (x, x), (x_{max}, x))$, respectively, $\forall x \in X$. As such, (x, x) is the common vertex of A and B. For example, consider the arbitrary point-cloud shown in Figure B2. According to Figure B2, the universe of discourse is as follows, $X = [20, 80]$. Notice the relative positions of the boxes denoted as A and B in Figure B2. The boxes are connected at their common vertices.

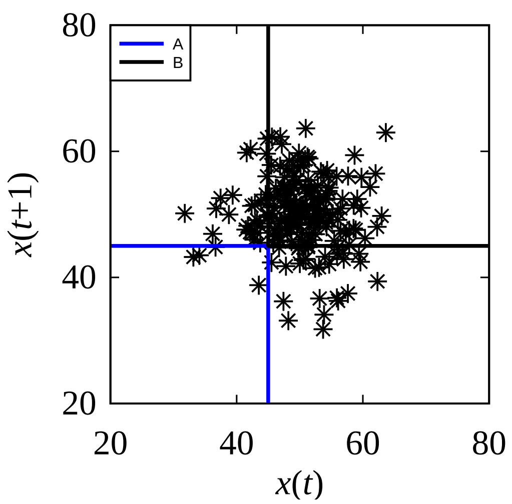

Figure B2. Relative position of A and B in the point-cloud ($x(t)$, $x(t+1)$).

Let $\Pr_A(x)$ and $\Pr_B(x)$ be two subjective probabilities, wherein $\Pr_A(x)$ and $\Pr_B(x)$ represent the degrees of chance that the points in the point-cloud are in A and B, respectively. As such, these functions are defined by the following mappings:

$$
\begin{aligned}
& X \to [0,1] \\
& x \mapsto \Pr_A(x) = \frac{\sum_{i=0}^{n-1} \Theta(t)}{n-1} \\
& \Theta(t) = \begin{cases} 1 & ((x(t) \le x) \wedge (x(t+1) \le x)) \\ 0 & otherwise \end{cases}
\end{aligned}
\quad \text{(B1)}
$$

$$
\begin{aligned}
& X \to [0,1] \\
& x \mapsto \Pr_B(x) = \frac{\sum_{i=0}^{n-1} \Omega(t)}{n-1} \\
& \Omega(t) = \begin{cases} 1 & ((x(t) \ge x) \wedge (x(t+1) \ge x)) \\ 0 & otherwise \end{cases}
\end{aligned}
\quad \text{(B2)}
$$

The typical natures of the functions defined in Equations (B1) and (B2) are illustrated in Figure B3, using the information of the point-cloud shown in Figure B2. Note that $\Pr_A(x)$ increases with the increase in x, and the opposite is true for $\Pr_B(x)$. It is worth mentioning that $\Pr_A(x) + \Pr_B(x) \le 1$ for the point-cloud, though for some cases, $\Pr_A(x) + \Pr_B(x) = 1$ (see Figure B4). This means that the expression $\Pr_A(x) + \Pr_B(x)$ does not serve the role of "cumulative probability distribution." A cumulative probability distribution can, however, be formulated by using the information of $\Pr_A(x)$ and $\Pr_B(x)$, as shown in Figure B4.

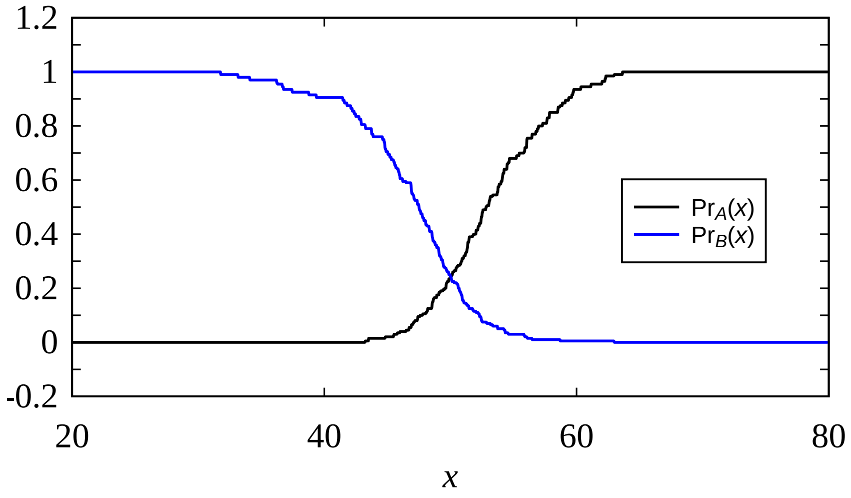

Figure B3. The typical nature of $\Pr_A(x)$ and $\Pr_B(x)$ for unimodal quantity.

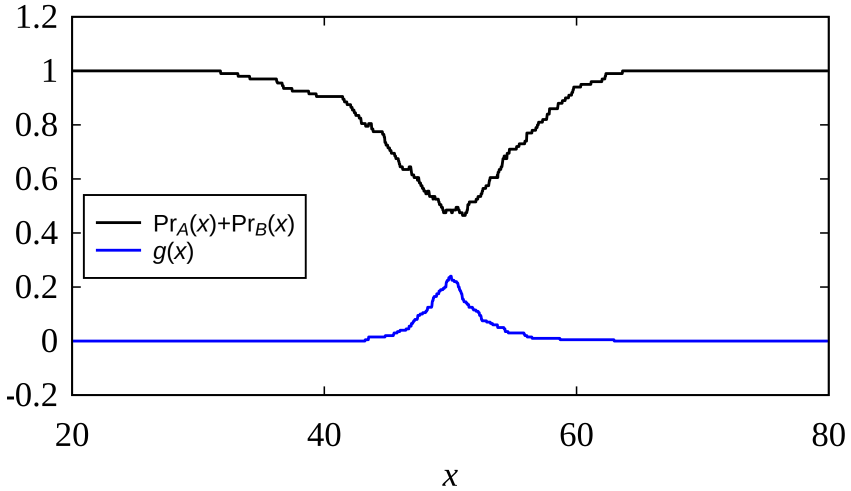

Figure B4. Nature of $Pr_A(x) + Pr_B(x)$ and $\min(Pr_A(x), Pr_B(x))$ for unimodal data.

Consider a mapping that maps x into the minimum of $Pr_A(x)$ and $Pr_B(x)$, as follows:

$$\begin{aligned} &X \to [0, a] \\ &x \mapsto g(x) = \min(\Pr_A(x), \Pr_B(x)) \end{aligned} \tag{B3}$$

In Equation (B3), $a = 1$ if the point-cloud is a point; otherwise, $a < 1$. Figure B4 shows the nature of $g(x)$ with respect to $Pr_A(x) + Pr_B(x)$. The area under $g(x)$ is given by:

$$Q = \int_X g(x)dx \tag{B4}$$

There is no guarantee that $Q = 1$. Otherwise, $g(x)$ could have been considered a probability distribution of the underlying point-cloud. However, a function $F(x)$ can be defined as follows:

$$\begin{aligned} &[0, a] \to [0,1] \\ &x \mapsto F(x) = \frac{\int_{x_{\min}}^{x} g(x)dx}{Q} \end{aligned} \tag{B5}$$

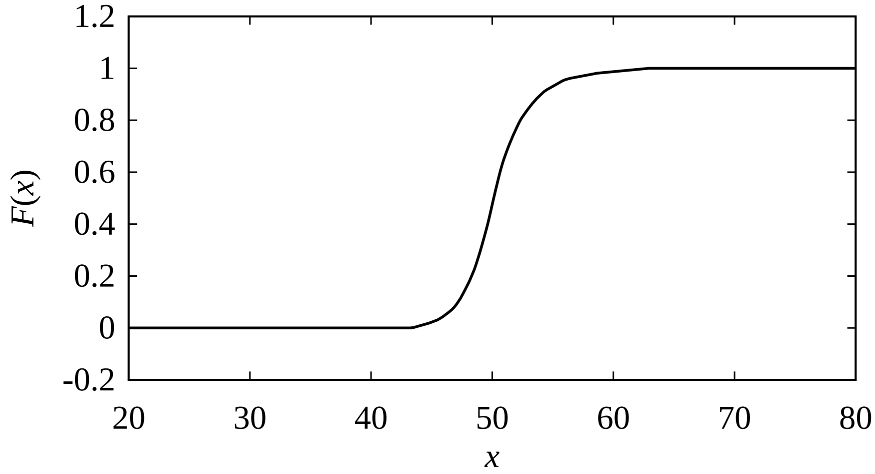

Figure B5. Nature of cumulative probability distribution of a point-cloud.

$F(x)$ can be considered a cumulative probability distribution because $\max(F(x)) = 1$, $F(x) \geq F(z)$ for $x \geq z$, $F(x) \in [0, 1]$, $\forall x, z \in X$. Figure B5 shows the nature of $F(x)$ derived from $g(x)$ shown in Figure B4. The cumulative probability distribution defined in Equation (B5) produces a probability distribution $\Pr(x)$. Thus, the following formulation holds:

$$\Pr(x) = \frac{dF(x)}{dx} \tag{B6}$$

Figure B6 shows the probability distribution $\Pr(x)$ that corresponds to $F(x)$ as shown in Figure B5. The area under the probability distribution $\Pr(x)$ is unit and $\Pr(x)$ remains in the bound of [0, 1].

From the induced probability distribution $\Pr(x)$, a possibility distribution given by the membership function $\mu_I(x)$) can be defined based on the heuristic rule of probability-possibility transformation—the degree of possibility is greater than or equal to the degree of probability. The easiest formulation is to normalize $\Pr(x)$ by its maximum value, $\max(\Pr(x) \mid \forall x \in X)$, yielding the following formulation:

$$\begin{aligned} &[0,1] \to [0,1] \\ &\Pr(x) \mapsto \mu_I(x) = \frac{\Pr(x)}{\max(\Pr(x) \mid \forall x \in X)} \end{aligned} \tag{B7}$$

Figure B7 shows the possibility distribution $\mu_I(x)$ derived from the probability distribution $\Pr(x)$ shown in Figure B6. The shape of the induced probability and possibility distributions are identical, as evident from Figures B6 and B7, respectively. Other formulations can be used instead of the formulation (B7), if needed.

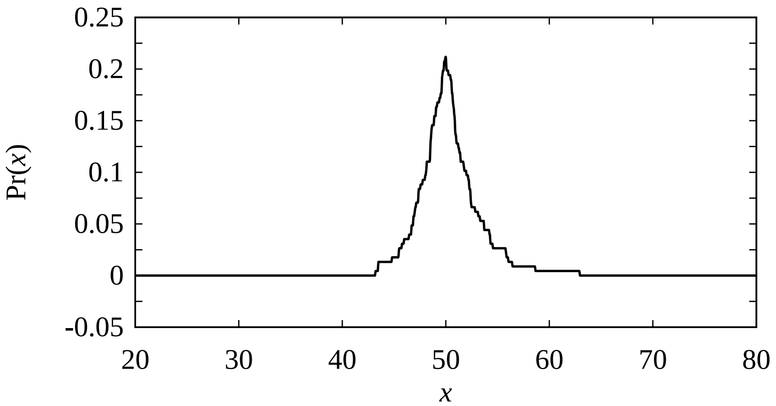

Figure B6. The nature of the probability distribution of a unimodal point-cloud.

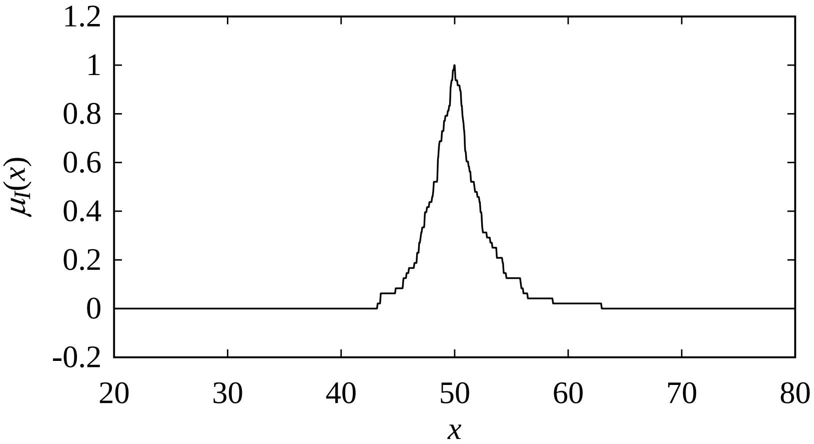

Figure B7. The nature of the possibility distribution of a unimodal point-cloud.

However, it is observed that when the point-cloud resembles the point-cloud of a bimodal quantity, the induced possibility distribution resembles a trapezoidal fuzzy number. In addition, when the point-cloud is a point, the induced possibility distribution becomes a fuzzy singleton. Moreover, when the point-cloud resembles the point-cloud of unimodal data, the induced probability/possibility distribution resembles a triangular fuzzy number. To define the membership function of an induced fuzzy number in the form of a triangular fuzzy number, the following formulation can be used.

Let u, v, and w be three points in ascending order in the universe of discourse X, $u \le v \le w \in X$. Let the interval $[u, w]$ be the *support* of a triangular fuzzy number and the point v be the *core*. The following procedure can be used to determine the values of u, v, and w from the induced fuzzy number $\mu_I(x)$ (Equation (B7)):

$$\begin{aligned}
& u \le v \le w \in X \\
& u = x \quad \left(\mu_I(x) = 0 \wedge \mu_I(x + dx) > 0\right) \\
& v = x \quad \left(\mu_I(x - dx) < 1 \wedge \mu_I(x) = 1\right) \\
& w = x \quad \left(\mu_I(x - dx) > 0 \wedge \mu_I(x) = 0\right)
\end{aligned} \tag{B8}$$

As defined in (B8), u is the point after which the membership value $\mu_I(x)$ is greater than zero, v is the point corresponding to the maximum membership value $\max(\mu_I(x))$, and w is the point from/beyond which the membership value $\mu_I(x)$ again becomes/remains zero. Thus, the membership function of the induced triangular fuzzy number denoted as $\mu_T(x)$ is as follows:

$$\begin{aligned}
& X \to [0,1] \\
& x \mapsto \mu_T(x) = \max\left(0, \min\left(\frac{x-u}{v-u}, \frac{w-x}{w-v}\right)\right)
\end{aligned} \tag{B9}$$