

Deterministic Design, Reliability-Based Design and Robust Design

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ABSTRACT

Due to the inherent uncertainties or variabilities in loads, materials and manufacturing quality, variabilities are unavoidable in structural responses. To ensure the reliability of a structure, these uncertainties or variabilities must be considered during structural design. Through a simple cantilever box beam example, the concepts and practices of three design methodologies: deterministic design, reliability-based design, and robust design, are examined in this paper. Particular attention is given to the meaning of robust design and its definition in the context of reliability-based design. Several robustness criteria are studied and proposed in an attempt to search for a proper objective function in a reliability-based design framework. The stress analysis is carried out using both MSC/NASTRAN and an analytical formula.

INTRODUCTION

Uncertainties or variabilities exist in loading, material properties, geometry and other aspects of any structure. Such uncertainties may be classified as reducible or irreducible. Reducible uncertainties are usually caused by lack of data, modeling simplifications, human errors, etc., and can be reduced through, among other things, collecting more data, better understanding of the problem, more strict quality control. Irreducible uncertainties are caused by phenomenon of a random nature and can not be reduced by possession of more knowledge or data.

Because of the existence of such uncertainties in the life cycle of a structure the structural response and life also show scatter. To design structures that can perform their intended function with desired confidence, the uncertainties involved must be taken into account. The traditional way of dealing with the uncertainties is to use conservative values of the uncertain quantities and/or safety factors in the framework of deterministic design. A more rigorous treatment of the uncertainties can be found in reliability-based design philosophies that have been under development for the last half of a century and are gaining more and more momentum. More recently, the concept of “robust design” has become very popular [1]. However, there is no universal agreement as to the meaning of “robustness,” let alone a quantifiable criterion. Is it an entirely new design philosophy or is it part of reliability-based design? Can reliability-based design be replaced by robust design? What are the advantages and disadvantages of each of these design methodologies? This paper investigates these questions by designing a simple cantilever beam with all three methods.

To facilitate the analysis, a general purpose reliability-based analysis computer program, PRODUCTS (PRobabilistic Optimum Design Under ConstraiNTS) [2], has been used for this study. PRODUCTS integrates a probabilistic analysis module, FPI [3], an optimizer, and MSC/NASTRAN for structural reliability-based design. The DSA (Design Sensitivity Analysis) capability [4] in MSC/NASTRAN is used to accelerate the probabilistic analysis process. Work is currently ongoing to verify PRODUCTS with both small and large FE models. As one of the verification examples, the example problem used in this paper has been solved by PRODUCTS using both an analytical solution and MSC/NASTRAN. The resulting designs are very close.

The example problem will be described in the next section. Then, it is designed using a deterministic design method, a reliability-based design method and a robust design method. These designs will be compared and analyzed, and some observation made at the end of the paper.

DESIGN OF A CANTILEVER BEAM

Figure 1 shows the cantilever beam with a rectangular cross-section subjected to a vertical load, P_y , and a horizontal load, P_x , at the tip. The design objective is to prevent

yielding due to bending stress while keeping the weight of the beam low, i.e. to design cross-sectional dimensions, w , h , and t such that

$$g \equiv R - S = R - \left[\left(\frac{lw}{2I_y} \right) P_x + \left(\frac{lh}{2I_x} \right) P_y \right] > 0 \quad (1)$$

where R is the yield strength and S the maximum bending stress. Assume that there are scatters in P_x , P_y , and R , and they follow normal distributions with the following parameters:

Table 1: Statistics of Loads and Material Properties

| | P_x (lbf) | P_y (lbf) | R (psi) |
|--------------------|-------------|-------------|-----------|
| Mean | 500 | 1000 | 40,000 |
| Standard Deviation | 150 | 50 | 2,000 |

It is assumed that the manufacturing tolerances on the cross-sectional width, height and wall thickness are relatively small and, therefore, the dimensions will be treated as deterministic design variables.

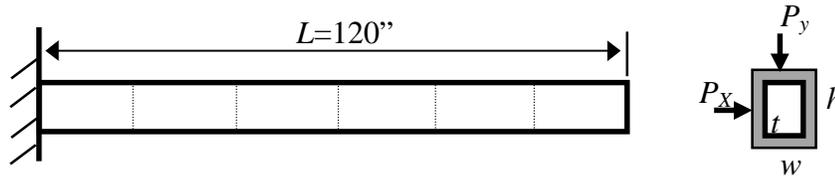


Figure 1: The Cantilever Beam (with the Finite Element Mesh)

DETERMINISTIC DESIGN (Design 1)

Depending on the discipline and design code adopted, different methodologies can be used. What is used here is the 95th percentile value for the loads and 5th percentile value for the material properties. That is,

$$\begin{aligned} P_x &= 500 + 1.645 \times 150 = 746.75 \text{ lbf} \\ P_y &= 1000 + 1.645 \times 50 = 1082.25 \text{ lbf} \\ R &= 40000 - 1.645 \times 2000 = 36710.0 \text{ lbf} \end{aligned}$$

Using these values, the design with minimum weight can be obtained by solving the following optimization problem:

$$\begin{aligned} \text{Minimize: Area} &= wh - (w - 2t)(h - 2t) \\ \text{Subjected to: } g &\geq 0 \\ &2'' \leq w, h \leq 10'' \quad 0.1'' \leq t \leq 0.5'' \end{aligned} \quad (2)$$

Note that since the beam length and density are constant, using the cross-sectional area as the objective function will yield the same answer as using the weight as the objective function.

Equation (2) is a typical nonlinear programming problem, and is here solved using PRODUCTS. The g-function is calculated using Eq. (1) as well as MSC/NASTRAN with 6 beam elements. The final design is listed in Column 2 of Table 2.

Table 2: Results of Different Design

| Designs | (Design 1) Determ. Design 95% rule | (Design 2) Min: Weight S.t.: $\beta \geq 3$ | (Design 3) Min: $\Sigma(d\beta_i)^2$ S.T. $\beta \geq 3$ weight ≤ 3 coeff. set 1 | (Design 4) Min: $\Sigma(d\beta_i)^2$ S.T. $\beta \geq 3$ weight ≤ 3 coeff. set 2 |
|--|---|---|---|---|
| w | 6.387 | 7.030 | 4.352 | 10.000 |
| h | 7.267 | 6.909 | 10.000 | 4.633 |
| t | 0.100 | 0.100 | 0.106 | 0.104 |
| Safety Index | 2.591 | 3.000 | 2.993 | 2.987 |
| Area | 2.691 | 2.748 | 2.998 | 3.000 |
| Weighted total sensitivity (all $w_i's=1$) | 21.56 | 17.80 | 13.09 | 23.88 |
| Weighted total sensitivity ($w_1 \sim w_3=0$) | 0.520 | 0.490 | 0.601 | 0.404 |
| $(\mu_{P_x}/\beta)(d\beta/d\mu_{P_x})$ | -1.049 | -0.880 | -0.959 | -0.781 |
| $(\mu_{P_y}/\beta)(d\beta/d\mu_{P_y})$ | -1.962 | -1.777 | -1.217 | -2.371 |
| $(\mu_R/\beta)(d\beta/d\mu_R)$ | 4.011 | 3.658 | 3.176 | 4.152 |
| $(\sigma_{P_x}/\beta)(d\beta/d\sigma_{P_x})$ | -0.665 | -0.628 | -0.741 | -0.490 |
| $(\sigma_{P_y}/\beta)(d\beta/d\sigma_{P_y})$ | -0.065 | -0.071 | -0.033 | -0.125 |
| $(\sigma_R/\beta)(d\beta/d\sigma_R)$ | -0.270 | -0.301 | -0.226 | -0.385 |
| μ_g | 9971.7 | 10936 | 12596 | 9633.5 |
| σ_g | 3848.7 | 3645.5 | 4208.8 | 3225.0 |

RELIABILITY-BASED DESIGN (Design 2)

In reliability-based design, g in Eq. (1) is called the limit state function or failure function. $g = 0$ divides the design space into two regions, the safety region ($g > 0$) and the failure region ($g < 0$). Because of the uncertainties in loads and yield strength, g is a random variable itself. As a result, we can not be certain in advance whether g falls into the safe region or the failure region for an arbitrary beam to be manufactured. We can only hope

that the beam is designed such that the probability that g is positive is sufficiently high. In mathematical terms, this is expressed as:

$$\text{Reliability} \equiv \text{Prob} [g > 0] \geq \text{Target Reliability} \quad (3)$$

In engineering practice, the safety index, β , instead of structural reliability is often used to represent the reliability level. When g has a normal distribution, β has a one-to-one correspondence with the structural reliability, given by

$$\beta = -\Phi^{-1}(1 - \text{Reliability}) = \frac{\mu_g}{\sigma_g} \quad (4)$$

where μ_g and σ_g are the mean and standard deviation of the g -function and Φ is the cumulative distribution function for the standard normal distribution. In the case where g has other distributions, Eq. (4) is not valid, but in general a larger β corresponds to a higher reliability level.

Depending on the goal of the design, different formulations can be used to achieve the design objective. For example, if the goal is to achieve maximum reliability as long as the weight is within some bounds, the design requirement can be expressed as:

$$\begin{aligned} \text{Maximize: } & \beta \\ \text{Subjected to: } & wh - (w - 2t)(h - 2t) \leq \text{Area upper bound} \\ & 2'' \leq w, h \leq 10'' \quad 0.1'' \leq t \leq 0.5'' \end{aligned} \quad (5)$$

If the concern is with the weight, the design can be formulated as:

$$\begin{aligned} \text{Minimize: } & \text{Area} = wh - (w - 2t)(h - 2t) \\ \text{Subjected to: } & \beta \geq \beta_T = 3 \\ & 2'' \leq w, h \leq 10'' \quad 0.1'' \leq t \leq 0.5'' \end{aligned} \quad (6)$$

The selection of a target safety index, β_T , is problem dependent and often controversial. A commonly used value is 3, corresponding to, for a normally distributed g , a reliability of 0.99865 or a probability of failure of 0.00135.

Equation (6) is used in this study to design the beam. The β value is computed using Eq. (4) for a given design. Again, Eq. (6) is solved using PRODUCTS, and the final design is given in Column 3 of Table 2. Note that in the case where g is not a normal random variable β can be computed using a probabilistic analysis method [5].

ROBUST DESIGN (Design 3 & 4)

Recently, “robust design” has become a popular design philosophy among major manufacturers. Unfortunately, there are many different opinions as to the meaning of robust design. A commonly used definition states that *a robust design is a design that is insensitive (or less sensitive) to input variations*. In other words, the best design is one which performs as expected in the face of both expected and unexpected variations; and it does so by virtue of the fact that the design is inherently insensitive to changes in the design parameters and service environment. Based on this definition, regions in the design space should be sought where the sensitivity of the important response quantities with respect to the key input variables is low (or ideally zero). While it is an attractive and powerful design concept, in practice it may be difficult (if not impossible) to achieve. For example, designs that are insensitive to all variations may be overly conservative and costly. Also, it should be remembered that designs which are not sensitive to key random variables cannot be “improved” by making changes to the mean values of those variables. This design characteristic may, in fact, be undesirable in many situations.

In practice, what we are really concerned with is making sure that expected variations do not result in unacceptable performance; and among such designs the most desirable design is the one that is least sensitive to unexpected variation caused, for instance, by unintended use of the product or lack of knowledge of the uncertainties. Therefore, a more practical definition for robustness may be that *a robust design is a design whose performance is not unacceptably compromised by expected variations in parameters which are known to effect its performance, and is more tolerant to unexpected variations*.

Both robust design and reliability-based design try to deal with uncertainties. Their differences and similarities are difficult and probably unnecessary to describe, since there is no universal agreement on the definition and practice of the robust design philosophy, and the range of reliability theory is ever expanding. The more important issue is to identify and combine the merits in both concepts.

In reliability-based design, all uncertain quantities are modeled as random variables (or processes if variation in time is important). *If the statistical distributions of the input random variables are well established (i.e. when all uncertainties or variabilities are reducible (or expected)) then all of the uncertainties have been counted for in the design process and the result of reliability-based design would be robust* by the more practical definition of robustness.

When the distributions of the input random variables contain uncertainty due, for example, to lack of data or unintended usage, the safety index or computed reliability will be subject to error. For example, if the mean value and standard deviations of the horizontal force, P_x , was obtained with only six samples, we would suspect that they may not be the true means and standard deviations of the force. When more samples become available, we may find that the mean value is actually 550 and the standard deviation 100. The question is how to ensure the robustness of the reliability-based design when

distributions of the random variables contain uncertainties. One method in reliability theory to address such problems is to model the input uncertainties using random variables with random means and/or standard deviations. The problem with this method is that another layer of uncertainties may be introduced when defining the distribution of the mean of a random variable. Instead of using this approach in this study, we will look into the uncertainties in the means and standard deviations of the input random parameters by using the concept of robust design.

In the context of reliability-based design, a definition of robustness that can be translated directly into a design criterion is that a robust design is one that is least sensitive to the change in the statistics of the input random variables (such as the mean, standard deviation and type of a distribution) within acceptable range of cost. We will call the design philosophy based on such a definition reliability-based robust design (RBRD).

Since structural design usually involves a number of input random variables, there are many sensitivity factors. Simultaneously minimizing all factors requires multi-objective optimization techniques. To simplify the problem, a weighted average of the sensitivity factors can be used as a single objective function. Using this objective, a reliability-based robust design criterion can be formulated as:

$$\begin{aligned}
 \text{Minimize: } & \sum_{i=1}^n w_i \left(\frac{\theta_i}{\beta} \frac{\partial \beta}{\partial \theta_i} \right)^2 \\
 \text{S. T.:} & \beta \geq \beta_T \\
 & \text{Cost} \leq \text{Budget}
 \end{aligned} \tag{7}$$

where w_i 's are the weight coefficient, and θ_i 's are the statistics of the random variables such as the means and standard deviations. The factor θ_i/β in front of each derivative is used to make them non-dimensional. With this multiplier, each sensitivity factor represents the percentage change in β for each percent of change in θ_i .

The final design will, in general, be sensitive to the weighting coefficients. The selection of appropriate weighting coefficients is problem dependent and deserves more study. In general, the weighting coefficients should be associated with the likelihood and magnitude of the potential change in the corresponding parameter. The greater the magnitude and likelihood a parameter might change, the larger the weighting coefficient should be.

Let us now consider the beam example problem. We will use 3 as the target reliability level (β_T), and 3 as the upper bound on the weight (cost). Now the question is how to select the weight coefficients. We will try two sets of coefficients and observe the results. Let the first set of coefficients be unity for all w_i 's. Suppose we are quite confident in the mean values of the three random variables, but are not sure about their standard deviations. We can make $w_1=w_2=w_3=0$ and $w_4=w_5=w_6=1$. Using these two sets of

weight coefficients, the final designs and their sensitivities are obtained and listed, respectively, in the last two columns in Table 2. It can be seen that these two design are drastically different. Actually, the first design emphasizes the mean values of the loads and the other design places emphasis on the variances of the loads. In reality, which design is more robust depends on whether the mean values or the variances of the random variables have greater variation.

RESULTS AND DISCUSSION

Table 2 summarizes the four designs of the beam. The deterministic design using 95th percentile values for loads and 5th percentile values for material properties yields a design with a safety index of 2.60, which is unacceptable. However, if we use 99th percentile values for load and 1st percentile values for material, we will get a safe design ($\beta = 3.75$), with penalty of higher weight (3.80). In general, deterministic design is capable of yielding safe designs if sufficiently large conservatism is built in. It, however, is difficult to achieve an optimal balance between the safety and economics using this approach.

Column 3 in Table 2 describes the reliability-based design. It is the lightest design possible with a reliability level corresponding to a safety index of over 3. For the case where the parameters of the input random variables such as means and standard deviations are well established, and unexpected variations are not of great concern, the reliability-based design is also a robust design because all the important uncertainties have been accounted for.

When the statistics of the input random variables are subject to uncertainty, the reliability result contains uncertainty, and the reliability-based design may not be the most robust design. To achieve a design that satisfies the required reliability level and at the same time minimizes the performance reduction caused by the potential changes in the distributions of the input random parameters, reliability-based robust design approach can be employed. Two designs based on Eq. (7) are listed in Table 2.

Figure 2 shows the probability density functions for the above mentioned four designs. Figure 3 shows the normalized sensitivity factors of these four designs. Designs 2, 3, & 4 all have $\beta = 3$, but the g-function has different means and standard deviations. Design 2 has the minimum weight, but the largest sensitivities. When there is little or no uncertainty in the means and standard deviations of the input random variables, Design 2 is most desirable. Designs 3 and 4 have the same reliability and weight, but different sensitivity factors.

Design 3 is obtained by minimizing the reliability sensitivities with equal weight for all parameters. Since the sensitivity factors with respect to the means are much larger than those with respect to the variances, this design is pushed furthest to the right. Design 4 is obtained by minimizing the reliability sensitivities with respect to only the variances of the random variables, its g-function has the smallest variance. When there are no

variations in the means of the random variables, this design should be the most robust one.

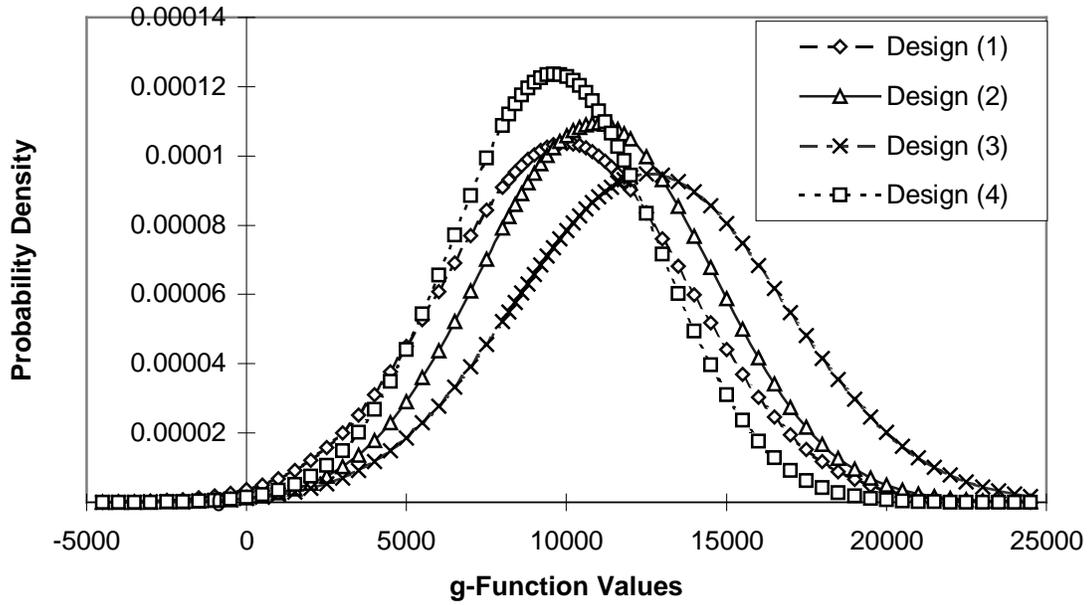


Fig. 2 PDF of g-Function for Different Designs

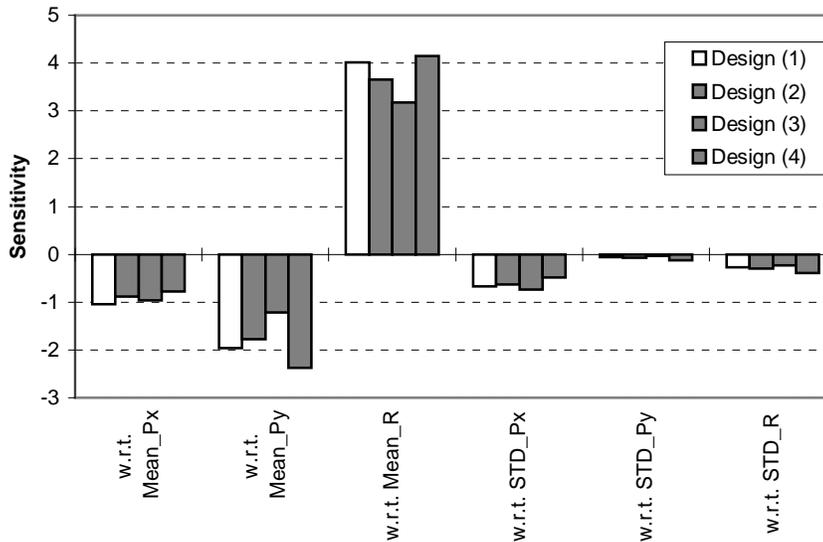


Fig. 3 Normalized Sensitivity Factors for Different Design

CONCLUSIONS

- (1) The reliability-based design is the most robust and economical design when the statistics (such as the mean and standard deviation) of the input random variables are well defined.
- (2) In the context of reliability-based design, robustness can be defined in terms of reliability being least sensitive to variations in the statistics (such as means and standard deviations) of the random variables while still meeting design cost and performance targets.
- (3) When the uncertainties in the statistical distribution of the random variables are thought to be significant, reliability-based robust design provides a useful design tool to minimize the impact of such uncertainties.
- (4) The design that minimizes the weighted average of the reliability sensitivity factors can be highly dependent on the weighting coefficients selected. In general, the weight coefficients must be selected to reflect the uncertainties in the distributions of the random variables.

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