ON THE TREE-AREA RATIO AND CERTAIN OF ITS APPLICATIONS

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It is the purpose of this paper to develop and illustrate what may be called a stocking norm, or tree-area ratio, for a given condition of timber stand and type; such as upland hardwood in the Piedmont Plateau, which is mixed in species and very uneven-aged, or loblolly pine in normal even-aged stands. The method is based upon the allocation of ground area to tree size.

Investigation of the same average d.b.h.

Both of these expressions of density of stocking have well-known drawbacks. Basal area to the acre is a number which, in itself, conveys little notion as to its make-up in frequency of trees according to size; likewise, the number of trees in a stand may be associated with an entirely different distribution of diameters than that of a normal stand having the same average.

Roth's spacing figure, D/d, described by Matthews³ is a useful rule of thumb for purposes of estimating density of stocking. In this, the square of the numerator is the average ground area used by the *n* trees of a stand of area A: that is,

$$D^2 = \frac{A}{n}.$$

The symbol d stands for the d.b.h. of average basal area. The spacing figure presupposes, therefore, that the ground area allocated to the tree is proportional only to the square of d.b.h., and it provides for no differentiation of area allocation according to species groups of the same stand, or according to size among trees of the same species.

None of these devices apply—as, indeed, they were not meant to apply—to uneven-aged stands.

Тне Метнор

If the ground area, Y, of a tree in the forest may be expressed as a function of its d.b.h., d, such that

$$Y = b_0 + b_1 d + b_2 d^2$$

where b_0 , b_1 , and b_2 are constants; then the total ground area distributed among all the trees on an acre, or sample plot, is the sum of the areas allocated to the individual trees thereon. If there are *n* trees

$${}^{n}_{S(Y)} = b_{0} [n] + b_{1} [{}^{n}_{S(d)}] + b_{2} [{}^{n}_{S(d^{2})}] \dots \dots (1)$$

in which $\overset{n}{S}$ denotes the sum over all n trees of the parenthesized quantities following it.

Now we cannot write out observation equations for each tree of the sample plot because we cannot allocate the ground area occupied by individual trees with satisfactory approximation. But unquestionably we may write observation equations of the form of (1); for if n

 $\overset{n}{S}(y)$ be taken as the *actual* area of a sample plot, representative of the timber upon which

the stocking norm is to be based, then $n, \overset{n}{S(d)}, n$

and $\tilde{S}(d^2)$ are, respectively, the number of trees, the sum of their diameters, and the sum of their squared diameters. Given such plot values for Nseparate and independent plots, the constants of equation (1) may be calculated by the method of least squares.

In conformity with the usual practice, the observations of each plot may be expressed on unitarea basis, such as an acre, and given unit weight. Notation may then be considerably

¹The writers extend their gratitude to Miss Lucille Parker, of the forest mensuration laboratory, Duke University, for her splendid help in carrying out the many painstaking calculations required by the investigation.

²Reineke, L. H. Perfecting a stand-density index for even-aged forests, Jour. Agric. Research. 46: 627-638. 1933.

⁸Matthews, D. M. Management of American forests. McGraw-Hill, New York. 495 pp. 1935.

simplified if we define-

- y = 1, the plot area, say, an acre x_0 , the number of trees to the acre

 - x_1 , the sum of diameters (d.b.h. in inches) to the acre x_2 , the sum of squared diameters to the acre

The equation then takes the form

 $Y = b_0 x_0 + b_1 x_1 + b_2 x_2 \dots$(2) in which Y is the calculated area used by the trees of a plot of given x_0 , x_1 , and x_2 , the plot area being unity.

In order to calculate the numerical values of the coefficients b_0 , b_1 , and b_2 , the sum of squares of residuals over the N separate and independent plots, that is,

$$S(1 - b_0 x_0 - b_1 x_1 - b_2 x_2)^2$$

N

is to be minimum. Upon differentiating this expression with respect to each of the coefficients in turn, the resulting normal equations are the following:⁴

$$\begin{array}{ccccc} N & N & N & N & N \\ b_0 S(x_0^2) & + b_1 S(x_0 x_1) & + b_2 S(x_0 x_2) & = S(x_0) \\ N & N & N & N \\ b_0 S(x_0 x_1) & + b_1 S(x_1^2) & + b_2 S(x_1 x_2) & = S(x_1) \\ N & N & N & N \\ b_0 S(x_0 x_2) & + b_1 S(x_1 x_2) & + b_2 S(x_2^2) & = S(x_2) \end{array}$$

It is to be noted that the sum of squares of relative residuals is minimized in this equation type, that is, the sum of squares of the ratios of residual to actual ground area.

The equation therefore expresses the ratio of the calculated ground area to corresponding actual ground area (taken as unity). A ratio in this form is, in practice, an unequivocal expression of density of stocking, and a commonsense one. Incidentally, equation (2) contains the average ground area allocated to single trees according to d.b.h., d; for by putting n=1 for x_0 , d and d^2 for x_1 and x_2 , respectively, we have $Y = b_0 + b_1 d + b_2 d^2$

where Y is the proportional part of unit plot area allocated the single tree of diameter d. A pictorial representation of this equation should have considerable silvicultural value in itself.

APPLICATION TO FULLY STOCKED LOBLOLLY PINE

The equation type was first applied to the data of 133 sample plots of fully stocked loblolly pine⁵ used in the construction of normal yield

tables.⁶ For each plot the record contains the number of trees to the acre according to 1-inch d.b.h. classes, starting with the one-inch class; and the basal area to the acre, as derived from the d.b.h. distribution. These were expressed on the mil-acre basis to facilitate computation. Then x_0 of equation (2) is the number of trees to the milacre; x_1 is the sum of products of d.b.h. and tree frequency in the diameter class, also to the milacre; and x_2 is the basal area to the mil-acre in square feet after dividing by the constant $\pi/4$ (144) or 0.005454 since

$${}^{n}_{S(d^2)} = \frac{\text{Basal area in square feet}}{0.005454}$$

where d is in inches.

The normal equations derived from the tabulation of these data are the following:

whence

 $b_0 = 0.0480; \ b_1 = 0.0668; \ b_2 = 0.0267$

The ground area allocation of a single tree in terms of its diameter, d, is then

 $Y = 0.0480 + 0.0668d + 0.0267d^2$

where Y is in mil-acres, and d is in inches.

When, as in the present case, the equation is based upon the data of fully stocked plots, it





^eU. S. Dept. Agric. Volume, yield and stand tables for second-growth southern pines. Misc. Pub. No. 50. 1929.

^{&#}x27;This equation type was used effectively by Day in a logging cost study. See, Day, B. B. A suggested method for allocating logging costs to log sizes. Jour. of Forestry. 35: 69-71, 1937.

[&]quot;The authors are indebted to the director of the Appalachian Forest Experiment Station for the use of these data.

expresses the area allocation, according to d.b.h., in fully stocked stands. If applied to *plot* data it expresses the tree-area ratio, that is, the ground area allocated to the trees of the plot as a ratio of the plot area. Applied to the single tree, its pictorial representation is presented in Figure 1A.

Applying it to the plot, we take as an example one of age 16 years, and site index 95 feet at 50 years, on which, to the mil-acre.

,,,			
number of trees	=	0.216	
sum of diameters	=	2.168	
sum of squared diameters	\equiv	23.048	
ing these charmetions into the			

Putting these observations into the equation, we find that the area used by the trees of the plot is

Y

$$= 0.0460(0.216) + 0.0668(2.168) + 0.0267$$

(23.048) = 0.770

to the mil-acre; or using the tree-area ratio as a measure of stocking this plot is 77 percent of average (and in this case of normal) density of stocking.

The calculated curve of Figure 1A is not without sampling error. The standard error⁷ of the calculated Y in terms of d is shown in Figure 1B. It is minimum for trees of about 6 inches d.b.h.

INFLUENCE OF AGE AND SITE INDEX

In even-aged, fully stocked stands, the part that age and site index may play in the treearea ratio is, perhaps, somewhat complex. It can hardly be of the same magnitude for all tree sizes encountered. One might expect, rather, some *joint* effect of age, site, and tree-size on the tree-area ratio; and it may, or may not, be strictly proportional to d.b.h. In line with these possibilities, the effect of age and site index may be calculated by considering the coefficients b_0 , b_1 , and b_2 of equation (2) in which

$$Y = b_0 x_0 + b_1 x_1 + b_2 x_2 \dots (2)$$

to be dependent upon age, A, and site index, S. Taking linear functions as first approximations, let

$$b_{0} = g_{0} + g_{1} A + g_{2} S$$

$$b_{1} = h_{0} + h_{1} A + h_{2} S$$

$$b_{2} = k_{0} + k_{1} A + k_{2} S$$
(3)

in which the numerical equivalents of the coefficients g, h, and k are to be determined after defining new variables by putting (3) into equation (2). This latter operation gives

$$Y = (g_0 + g_1A + g_2S)x_0 + (h_0 + h_1A + h_2S)x_1 + (k_0 + k_1A + k_2S)x_2.$$

The new variables are defined upon carrying

through the products indicated; that is

 $\begin{array}{l} Y = g_0(x_0) + g_1(Ax_0) + g_2(Sx_0) + h_0(x_1) + h_1\\ (Ax_1) + h_2(Sx_1) + k_0(x_2) + k_1(Ax_2) + k_2(Sx_2) \dots (4)\\ \text{for which the independent variables are now}\\ \text{parenthesized.} \end{array}$

There are nine of them. Three are the same as used in equation (2) in which the possible influence of age and site was neglected. The remaining six are due to the introduction of age and site index. If the contribution of these six to the sum of squares eliminated by fitting equation (4) to the 133 loblolly pine sample plots at hand is significant, then age and site index influence the tree-area ratio of these data perceptibly.

The test is carried out in Table 1. The sum of squares in the first line is that portion of the total sum of squares which is accounted for by fitting equation (2) for which any influence of age and site index was entirely disregarded. The sum of squares in the third line is that portion of the total sum of squares accounted for by equation (4), and it contains the influence of age and site index on the tree-area ratio. The difference between these two sums of squares is then the contribution of age and site index. Its mean square, 0.0314, is so nearly the same as the mean square of the residuals, 0.0392, that the evidence is conclusive that for the loblolly pine data at hand, there is no perceptible effect of age and site index on tree-area ratio-hence on the ground area allocation of a tree of given d.b.h.

Application to the Pine-Hardwood Type of the Piedmont Plateau

An ecological query with practical silvicultural implications has to do with the relative ground area used by trees of certain species groups in the many-aged pine-hardwood types of the Piedmont Plateau of the south Atlantic states. These interesting timber types are regarded as a stage in succession between the pure, or nearly pure, pine types which characterize the early decades following abandonment of cultivation and the many-aged, pure hardwood of the climax.

The problem was attacked by extending the tree-area ratio equation simultaneously to the groups of species into which the composition of the pine-hardwood types were classified. At most, four groups were to be segregated; namely, pine, overstory hardwood, dogwood, and remaining understory. If we symbolize these as P,

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⁷Fisher, R. A. Statistical methods for research workers. (Sec. 26 and 29). Oliver and Boyd, Edinburgh. 6th ed. 1936.

Due to	Degrees of freedom	Sum of squares	Mean square
Regression on x_0 , x_1 , and x_2 Added effect of remaining	3	127 .9 532	
variables Regression on the nine	6	0.1887	0.0314
variables	9	128.1419	
Residual	124	4.8581	0.0392
Total	133	133.0000	

TABLE 1.—ANALYSIS OF VARIANCE OF LOBLOLLY PINE TREE-AREA RATIO

O, D, and U; and if, as before, subscripts 0, 1, and 2 represent, respectively, the number of trees to the unit area of plot, the sum of their diameters, and the sum of their squared diameters; then we have, at most, 12 independent variables as follows:

For the pine	P_0, P_1, P_2
For the overstory hardwood	O_0, O_1, O_2
For the dogwood	D_0, D_1, D_2
For the remaining understory	U_0, U_1, U_2

The tree-area ratio equation then takes the form

$$Y = p_0 P_0 + p_1 P_1 + p_2 P_2 + o_0 O_0 + o_1 O_1 + o_2 O_2 + d_0 D_0 + d_1 D_1 + d_2 D_2 + u_0 U_0 + u_1 U_1 + u_2 U_2$$

in which Y is the calculated area used by the trees of a plot; and the coefficients, denoted by lower-case letters with their subscripts, may be determined by the method of least squares.

The field work was confined to the pine-hardwood types of the Duke Forest of Durham and Orange counties, N. C. Fifty half-acre plots, which satisfied certain pre-determined specifications, were selected on the ground. These specifications, designed to render judgment of normal stocking as objective as possible, are the following:

1. None of the open-site, herbaceous vegetation should be present as part of the ground cover.

2. There should be no individual openings in the canopy greater than 2 mil-acres in crosssectional areas.

3. There should be no apparent interruption of the diameter distribution from that of the characteristic J-shape of many-aged stands.

4. Not more than one soil series should fall within plot boundaries.

Measurements were taken on only two of the five square chains of each plot, selected strictly at random. D.b.h. of stems above $41/_2$ feet in height were tallied according to species and 1-inch diameter class, with the exception of the

first class, the class interval of which is from zero to $\frac{1}{2}$ -inch. Stems in this class were tallied as $\frac{1}{4}$ -inch. Up to the 3-inch class, diameters were estimated by eye, but thereafter all measurements were made with the diameter tape.

Species groups of special interest were the four as listed above. The particular interest in dogwood lies in the supposition that its space demands are unusually great; and its occurrence is sufficiently abundant to test the supposition.

Classification of nearly 50 species of hardwood in the overstory and understory groups turned out, naturally, to be a problem of considerable debate. The grouping finally adopted for the types is the following:⁸

Pine

Pinus echinata Pinus taeda Pinus virginiana Overstory hardwoods Hicoria alba Hicoria glabra Hicoria ovata Liquidambar styraciflua Liriodendron tulipifera Ouercus alba Quercus borealis maxima Quercus coccinea Quercus rubra **Ouercus** stellata Quercus velutina Dogwood Cornus florida Understory Acer rubrum Aesculus neglecta georgiana Alnus rugosa Amelanchier canadensis Carpinus caroliniana Cercis canadensis Chionanthus virginica Crataegus spp. Diospyros virginiana Evonymous americanus Fagus grandifolia Fraxinus americana Fraxinus pennsylvanica Fraxinus pennsylvanica lanceolata Ilex opaca

⁸Nomenclature is taken from Korstian, C. F., and W. Maughan. The Duke forest, a demonstration and research laboratory. Duke Univ. Forestry Bull. 1. 74 pp. 1935.

Juniperus virginiana Magnolis tripetala Morus rubra Nyssa sylvatica Ostrya virginiana Oxydendrum arboreum Prunus serotina Quercus marilandica **Ouercus** phellos Rhodendron nudiflorum Rhus copallina Sassafras variifolium Styrax grandifolia Ulmus alata Viburnum affine Viburnum prunifolium Viburnum rufidulum

While the classification of the tree species encountered is based upon the hypothesis that trees of the same d.b.h. but of different groups occupy different areas, this hypothesis is subject to direct test, based upon the comparison of the residual sum of squares of plot area not accounted for by the calculated tree areas. Accordingly, the tree-area ratio equation was fitted to the plot data several times, using different combinations of the groups each time. Each fitting yielded, in consequence, a different sum of squares unaccounted for by the combination.

In the first fitting no differentiation of species was recognized, hence there were only three independent variables; number of trees, sum of diameters, and sum of squared diameters, for all groups combined. The sum of squares independent of the equation, 1.27690 out of the total of 50 (unity for each plot), is listed in the first line of Table 2 with the 47 degrees of freedom upon which it is based.

In the next fitting, the four groups were combined into two classes of two groups each, namely, pine and overstory hardwood in one class, and the dogwood and remaining understory in the other. Since each of these two classes uses three independent variables, there are six in both. The sum of squares of residuals is now 1.22650, which with its 44 degrees of freedom is listed in the second line of Table 2.

In the third fitting, the pine and the overstory hardwood were admitted as separate groups of three independent variables each, while dogwood and the remaining understory group were combined into a single class with three independent variables. The residual sum of squares, based upon 41 degrees of freedom, is listed in the third line of Table 2.

Finally, the equation was fitted simultaneously to the data of the four separate groups, using 3 independent variables each and 12 degrees of freedom in all. The residual sum of squares is listed in the bottom line of Table 2.

This tabulation contains the information concerning the significance of the differentation of tree areas among the groups. Tests were based upon the mean square independent of the individual groups, the value 0.01504 of the bottom line of Table 2. They are presented in Table 3.

The first line of Table 3 contains the test as to whether the upper crown-canopy groups, consisting of pine and overstory hardwood combined, should be differentiated from the lower crown-canopy groups with respect to area of trees of the same d.b.h. The entry is the difference between the numbers in the first and second lines of Table 2. The latter table shows that when no grouping is recognized the sum of squares independent of the fitting is 1.27690 on 47 degrees of freedom; whereas when the upper crown groups are distinguished from the lower, the residual sum of squares is 1.22650 on 44 degrees of freedom. The difference between these, which is due to the differentiation of canopy in these two classes, is 0.05040 on 3 degrees of freedom. But since its mean square, 0.01680, differs but slightly from the error mean square, 0.01504, the evidence does not sustain the hypothesis that differentiation of upper and lower canopy as *classes* need be made.

Between the two groups of the upper canopy class, however, the case is quite different, as attested by the information listed in the second line of Table 3. The values therein, which are the remainders between the second and third lines of Table 2, contrast the pines with the

Table 2.—Sums of Squares of Residuals About the Tree-Area Ratio Equations as Based upon Pertinent Combinations of the Groups, Pine (P), Overstory Hardwood (O), Docwood (D), and Remaining Understory (U).

Grouping	Degrees of freedom	Sum of squares	Mean square
$\begin{array}{c} (P + O + U + D) \\ (P + O), (U + D) \\ (P) (O), (U + D) \\ (P), (O), (U), (D), \end{array}$	47 44 41 38	1.27690 1.22650 0.60195 0.57395	0.01504

Test	Degrees of freedom	Sum of squares	Mean square
(P + O) versus (U + D P versus O) 3	0.05040	0.01680
	3	0.62455	0.20818 ¹
	3	0.02800	0.00933
	38	0.57395	0.01504

TABLE 3.—TESTS OF SIGNIFICANCE OF GROUPING EFFECTS

¹Highly significant.

overstory hardwoods. The mean square is some 13 times the error mean square and is highly significant. There is no doubt, therefore, that pine and overstory hardwood trees occupy different tree areas, size for size.

Within the lower canopy class, on the other hand, we cannot discover any differentiation in tree area between dogwood and the remaining understory. The mean square in the third line of Table 3—based upon the difference between the numbers in the third and fourth lines of Table 2—is, in fact, less than the error mean square. There seems no reason, therefore, further to segregate these two groups.

The important finding of this investigation is the significant differentiation, with respect to tree area, of the two groups, pine and overstory hardwood, which characteristically make up the upper canopy of the pine-hardwood types. While we have not tested whether we should distinguish the lower-canopy class from *each* of the groups of the upper canopy, nevertheless it seems best to keep it separate in view of the fact that it contains so many species of diverse behavior. The equation of greatest utility, then, is that one which is based upon the groups P, O, and (U + D). Separating it into its three components and expressing it in terms of the single tree of diameter, d, it is as follows:

For the pines	Y =	$2.622 - 0.5109d + 0.0349d^2$
hardwood	Y = -	$0.0421 + 0.1764d + 0.0259d^2$
For the understory	$Y \equiv$	$0.1881 - 0.1789d + 0.0812d^2$

The pictorial representation of each is presented in Figure 2. At first glance the differences portrayed are both illuminating and puzzling. The most marked divergence in tree area is between the overstory hardwood and the pine for trees greater than 4 inches d.b.h. Judging only by these curves one can barely escape the inference that pine is definitely on the way out. This conclusion becomes inevitable upon comparing the curve of loblolly pine tree area of the pure pine type (Fig. 1) with the pine tree areas of the pine-hardwood types. In the latter case pines are confined to half the areas portrayed in the former.

The greater tree area of small pines as compared to that of middle-size ones (Fig. 2) is not to be ascribed to the caprice of chance unless the sample of plot data is a most improbable one. Confidence in the data, however, and the common observation that in these types pine will generally reproduce only in openings, suggests an explanation.

One of the specifications upon which the suitability of the half-acre plots was based is the limitation of individual openings in the canopy to areas not greater than 2 mil-acres in crosssection. But it is not at all unlikely that there are individual patches of ground area, of several mil-acres each, beneath which root competition is, or had been, sufficiently negligible as to tolerate establishment of pine seedlings; that soil moisture and nutrients were available to them far in excess of their needs as seedlings; but that from the sapling stage on, they are handicapped to final extermination in the root competition for the favors of the soil as neighboring hardwoods encroach upon their domain.

In the dogwood and remaining understory class the small areas allocated to trees below 4 inches d.b.h. is as anticipated. It seems rather surprising, however, that larger trees of the understory group of species do not also run to



Fig. 2.—Tree area in the pine-hardwood types in fully stocked stands according to d.b.h. in each of three species groups into which the composition of the types is classified.

smaller areas than the overstory hardwood group. This anomaly is probably the effect of the incompatability encountered in assigning certain species, occasionally represented by a large tree, to the understory group. One is inclined to the opinion, *after the analyses*, that any species represented by a tree over 8 inches in d.b.h., should not have been assigned to the understory group; for, in fact, the data are not represented by a single tree larger than 8 inches d.b.h. of a species assigned to the understory group, which, if judged only by its place within its immediate environment, would not have been assigned to the overstory.

SUMMARY

A method of allocating tree area according to d.b.h. of individual trees by means of a quadratic equation fitted by the method of least squares to sample *plot* data is developed and illustrated. Plot data in number of trees, sum of diameters, and sum of squared diameters, which supply the observations, are expressed on the unit-area bases—or, better, the areas of plots in the field are constant and taken as unity. The calculated equation is then an expression of density of stocking as a proportion of the average density of stocking represented by the data. Consequently, if the data represent fully stocked plots the equation supplies a stocking norm for the timber type, or condition, upon which it is founded.

Two applications are exemplified. The first deals with a mensurational problem involving the calculation of the tree-area ratio or stocking norm for even-aged, fully stocked, pure, loblolly pine from the data used in the construction of normal-yield tables. The investigation brought out conclusively, though unexpectedly, that there is no perceptible effect of age or site index upon the tree-area ratio in the data.

The second application has to do with an ecological query concerning tree-area in the pinehardwood types of the Piedmont Plateau. It was determined that the pine trees over 4 inches in d.b.h. were occupying very much less area for their sizes than the hardwood trees, regardless of whether the latter is typified as overstory, dogwood, or remaining understory. The greater tree area of small pines, is, however, a significant fact and not to be accepted as chance effect. It is explained by the known inability of pine to become established in the pine-hardwood types except in localized patches which are relatively free of other root competition.

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CANADA'S FOREST INDUSTRIES DEPEND ON SPRUCE

CANADA'S forest industries, particularly the pulp and paper industry, depend to a large extent for their existence upon a continuous supply of spruce, according to the Dominion Forest Service, Department of Mines and Resources, Ottawa. Spruce forms 39 percent of the accessible standing timber of the Dominion, and 25 percent of the wood used annually for all purposes. It is the principal wood used in the manufacture of pulp and paper, and ranks second only to Douglas fir in Canadian lumber production.

Spruce is the most widely distributed of any kind of timber, with its range extending from the Atlantic to the Pacific. There are five species in Canada—black, white, red, Engleman and Sitka. Black spruce and white spruce occur in each province. Red spruce is confined to the Maritime Provinces and Quebec, Engleman spruce to the interior of British Columbia and western Alberta, and Sitka spruce to the coastal region in British Columbia. The Sitka spruce is known in Great Britain as silver spruce, and is the best wood known for aircraft construction as it is light and resilient and does not splinter or shatter easily with impact.

Spruce reproduction in Canada appears to be at a disadvantage in competition with other less valuable tree species, and investigations are now being conducted to find out the best means of securing adequate reproduction and of increasing the rate of growth of this important species.